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From the Editor

MACIEJ SENDŁAK *

ON CONDITIONALS: PREFACE

If you are reading this, there is a good chance that you are interested in conditionals. Also, depending on how deep your interest is, you may recognize the first sentence of this paragraph as an example of a conditional statement. If you did not recognize this, you should know that conditionals are complex expressions of the form "If *A*, then *C*" (formally, "$A > C$"). We often use them to indicate a connection between two states of affairs, expressed by the antecedent *A* (or if-clause) and a consequent *C* (or then-clause). For example: "If you ever lose your credit card, immediately inform your bank", "If there is an action, there is an equal and opposite reaction", "If the river were to rise another two feet, the subway system would be flooded". By asserting statements like these, one usually suggests a relationship between two states, such that one affects the other. In other words, the second somehow obtains under the condition of the first.

While the syntactical structure of conditionals may seem quite simple, their semantic and pragmatic consequences are hard to overestimate. The importance of conditionals is partly grounded in their commonness. Accordingly, many claim that these are useful (if not indispensable) tools for expressing our emotions or beliefs, as well as for acquiring and transferring knowledge (Nickerson, 2015; Williamson, 2016). Some believe that it is impossible to experience genuine grief or satisfaction without involving the use of conditionals (Byrne, 2014). Others argue that the capacity to perform conditional inferences is a hallmark of intelligence. It is safe to say that conditionals are essential to our intellectual life. This partly explains why they have become the subject of academic interest.

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At the same time, the complexity of this subject and the richness of the uses of such expressions explain why it is so difficult to discover a statement about conditionals that is both interesting and finds universal acceptance among theorists. Accordingly, the question of conditionals is a breeding ground for vibrant debates among philosophers, psychologists, and linguists (Bennett, 2003; Kratzer, 2012; Sanford, 1989).

The debates mentioned concern such fundamental questions as what the logical structure of conditionals is and how their taxonomy looks. In this respect, many claim that one should draw a line between at least two types: indicatives and subjunctives (or counterfactuals). This distinction is nicely illustrated by the contrast between two sentences with the same antecedents and consequences, but different moods:

(IND) If Oswald did not kill Kennedy, someone else did.

(SUB) If Oswald had not killed Kennedy, someone else would have.

The first sentence is true, while the second seems to be false. If, during the investigation of the assassination of Kennedy, it had been proved that Oswald did not kill Kennedy, then someone else must have killed him. In this case, we rely on the knowledge that Kennedy had been killed, and we enrich this knowledge with the information that Oswald was not the person that killed him. If that is the case, then someone else did. After all, given that Kennedy was killed, someone must be his killer. Contrary to IND, SUB is false. In such a scenario, we consider a situation in which Kennedy had not been killed in the first place. Putting aside conspiracies, it is safe to assume that if Oswald had not killed Kennedy, Kennedy would not have been assassinated (Adams, 1970).

This also results in the question of whether IND and SUB should be the subjects of a uniform analysis. Some respond to this positively, and argue for a unified analysis of indicative and subjunctive conditionals (e.g., Chisholm, 1946; Nolan, 2003; Stalnaker, 1968), while others recommend treating them differently (e.g., Jackson, 1979; Lewis, 1973; Mackie, 1973). One of questions that follows is what the semantic nature of conditionals is: Are they subject to truth-values or not? Assuming that truth-values are properties of propositions, views that share the assumption mentioned might be labeled "propositional approaches". Not every analysis of conditionals is like this. Some consider an assertion of a conditional a distinctive *speech act*. While this involves two propositions (one of which is supposed, while the other is asserted in a way that is qualified by the supposition), it is claimed that the conditional itself is not a proposition. Accordingly, it is claimed that the condition of assertion of "$A > C$" depends upon the probability of C, given A. Thus, an assertion of "$A > C$" is acceptable or justified only if the probability of C being the case under the assumption of A being the case is sufficiently high. This is further determined by conditional probability $\Pr(C|A)$, which is analyzed in terms of the absolute probability of $\Pr(A + C) \div \Pr(A)$ (Adams, 1965; 1975; Appiah, 1985; Edgington, 1986; 1995).

A different view that also falls into the category of non-propositional approaches is one according to which conditionals are "condensed or telescoped arguments" (Mackie, 1973, p. 69). This means that when one asserts "$A > C$", one in fact performs more complex argumentation in which A is one of the premises and C is its consequence. Naturally, the other premises are often merely silently assumed, and not explicitly stated. The reason that Mackie's view differs from propositional approaches is that arguments are neither true nor false. Thus, if conditionals are considered "telescoped" arguments, they too are neither true nor false.

Interestingly, both of the above approaches have their propositional counterparts. Thus, some have claimed that conditional "$A > C$" is true if $\Pr(C|A)$ is sufficiently high (or close to 1). An important assumption here is that $\Pr(A)$ is greater than zero. Otherwise, the outcome of $\Pr(C|A)$ would be undefined. In such cases, it is commonly stipulated that all conditionals of impossible antecedents are vacuously true (for an alternative approach, see Hájek, 2003; Leitgeb, 2012a; 2012b; McGee, 1994). Likewise, there are also propositional counterparts of Mackie's analysis. These are views that track back to the works Frank Plumpton Ramsey (1931), according to which "'If *p* then *q*' means that *q* is inferable from *p*, that is, of course from *p* together with certain facts and laws not stated but, in some way, indicated by the context" (Ramsey, 1931, p. 248). As the consequent of a conditional is somehow meant to be inferred (with the support of particular facts and laws) from the antecedent, this approach is sometimes labeled "inferentialism" or "support theory" (Bennett, 2003, p. 302). This was a point of interest for the two most prominent advocates of the truth-functional version of support theory—Nelson Goodman (1947) and Roderick Chisholm (1955). After years of bad press, we are witnessing a revival of inferentialism that is heavily grounded in empirical research (Douven, 2008; Krzyżanowska, Wenmackers, Douven, 2013).

What seems to be the most popular analysis of counterfactuals is the one delivered in terms of possible worlds semantics (Lewis, 1973; Stalnaker, 1968; Todd, 1964). By virtue of this approach, the truth-value of conditional $A > C$ depends upon the similarity between the actual worlds and a world where both A and C are true, compared to a world where A and $\sim C$ are true. Finally, there is a further view that has been partly motivated by the obstacles of possible world semantics, and which is based on truthmaker semantics (e.g., Embry, 2014; Fine, 2012). One such obstacle is the question of the truth value of counterpossibles, namely, counterfactuals with impossible antecedents. Popular examples of these are:

If whales were fish, they would have gills.

If whales were fish, they would not have gills.

If Kate squared the circle, mathematicians would be impressed.

If Kate squared the circle, mathematicians would not be impressed.

A standard possible worlds semantics has it that as there are no worlds where the antecedent of the above conditionals is true, all of them are considered vacuously true. Many considered this consequence to be questionable enough to seek an alternative approach. While truthmakers' semantics provide an analysis that distinguish false and true counterpossibles, it should be stressed that such analysis is also possible within the extended possible worlds semantics (e.g., Nolan, 1997). Furthermore, the question of whether an adequate theory ought to allow for non-vacuously-true counterpossibles is itself a subject of a debate (Berto, French, Priest, Ripley, 2018; Brogaard, Salerno, 2013; Sendłak, 2021; Williamson, 2018).

The above is merely a glance at the notion of conditionals. However, it should be clear that this is both a complex and intriguing notion. The present issue of *Studia Semiotyczne* addresses some of the questions mentioned above. We are happy to present a collection of papers that reflect the complexity of the subject of conditionals. Thus, the issue includes an article, *The Nature of Propositional Deduction—a Piagetian Perspective*, in which M. A. Winstanley addresses the question of the relationship between the logic and psychology of reasoning. He does this by comparing two dominant approaches to this subject matter, and eventually proposes a third one, which is directly inspired by the works of Jean Piaget.

An essential part of this issue is devoted to the semantics of conditionals. *A Probabilistic Truth-Condition Semantics for Indicative Conditionals* by Michał Sikorski proposes an approach designed to overcome some of the common obstacles or limitations of a probabilistic account of conditionals; one of them being the conditionals of embedded antecedents. As mentioned, a significant debate within the semantics of conditionals concerns the truth value of counterpossibles. We have two papers on this subject. The first one—*Against Vacuism* by Samuel Dickson—relies on the role of counterpossibles within the context of natural science and mathematics. Along with characterizing the mechanism that underpins the motivation for vacuism, Dickson argues that some counterpossibles are false. Whereas Dickson focuses on the role of the counterpossible in scientific inquiry, Felipe Morales Carbonell investigates the issue of counterpossibles from the point of view of the notion of subject matters. This makes his *Towards Subject Matters for Counterpossibles* a paper that puts together two intriguing subjects of semantics. Accordingly, Morales Carbonell compares two popular approaches to the subject matters—so-called way-based and atom-based—from the point of view of the question of counterpossibles, and shows how this affects the theoretical virtues of each of them.

While a vast part of the work is dedicated to semantics, it is difficult—if possible, at all— to omit the question of the pragmatics of conditionals (Moss, 2012; von Fintel, 2001). While pragmatics was a wastebasket of philosophy for many years (Carston, 2017, p. 453), it is clear nowadays that it plays a crucial role in understanding the nature of conditionals. Mariusz Popieluch in *Context-Indexed Counterfactual* addresses this, by combining both semantic and pragmatic features of conditionals. A result of this is his proposal to include a context factor within the semantics of counterfactuals.

REFERENCES

Adams, E. (1965). The Logic of Conditionals. *Inquiry: An Interdisciplinary Journal of Philosophy*, *8*(1–4), 166–197.

Adams, E. (1970). Subjunctive and Indicative Conditionals. *Foundations of Language*, *6*(1), 89–94.

Adams, E. (1975). *The Logic of Conditionals: An Application of Probability to Deductive Logic*. Dordrecht: Springer.

Appiah, A. K. (1985). *Assertion and Conditionals*. Cambridge: CUP.

Bennett, J. (2003). *A Philosophical Guide to Conditionals*. Oxford: OUP.

Berto, F., French, R., Priest, G., Ripley, D. (2018). Williamson on Counterpossibles. *Journal of Philosophical Logic*, *47*(4), 693–713.

Brogaard, B., Salerno J. (2013). Remarks on Counterpossibles. *Synthese*, *190*, 639–660.

Byrne, R. M. J. (2016). Counterfactual Thought. *Annual Review of Psychology*, *67*(1), 135–157.

Carston, R. (2017). Pragmatics and Semantics. In Y. Huang (Ed.), *The Oxford Handbook of Pragmatics* (pp. 453–472). Oxford: Oxford University Press.

Chisholm, R. M. (1946). The Contrary-to-Fact Conditional. *Mind*, *55*(220), 289–307.

Chisholm, R. M. (1955). Law Statements and Counterfactual Inference. *Analysis*, *15*(5), 97–105.

Douven, I. (2008). The Evidential Support Theory of Conditionals. *Synthese*, *164*(1), 19–44.

Edgington, D. (1986). Do Conditionals Have Truth Conditions? *Critica*, *18*(52), 3–39.

Edgington, D. (1995). On Conditionals. *Mind*, *104*(414), 235–329.

Embry, B. (2014). Counterfactuals Without Possible Worlds? A Difficulty for Fine's Exact Semantics for Counterfactuals. *Journal of Philosophy*, *111*(5), 276–287.

Fine, K. (2012). Counterfactuals Without Possible Worlds. *The Journal of Philosophy*, *109*(3), 221–245.

Goodman, N. (1947). The Problem of Counterfactual Conditionals. *The Journal of Philosophy*, *44*(5), 113–128.

Hájek, A. (2003). What Conditional Probability Could Not Be. *Synthese*, *137*(3), 273–323.

Jackson, F. (1979). On Assertion and Indicative Conditionals. *Philosophical Review*, *88*(4), 565–589.

Kratzer, A. (2012). *Modals and Conditionals: New and Revised Perspectives*. Oxford: Oxford University Press.

Krzyżanowska, K., Wenmackers, S., Douven, I. (2013). Inferential Conditionals and Evidentiality. *Journal of Logic, Language and Information*, *22*(3), 315–334.

Leitgeb, Hannes (2012a). A Probabilistic Semantics for Counterfactuals. Part A. *Review of Symbolic Logic*, *5*(1), 26–84.

Leitgeb, Hannes (2012b). A Probabilistic Semantics for Counterfactuals. Part B. *Review of Symbolic Logic*, *5*(1), 85–121.

Lewis, D. (1973). *Counterfactuals*. Oxford: Blackwell.

Mackie, J. L. (1973). *Truth, Probability, and Paradox*. Oxford: OUP.

McGee, V. (1994). Learning the Impossible. In E. Eells, B. Skyrms (Eds.), *Probability and Conditionals: Belief Revision and Rational Decision* (pp. 179–199). Cambridge: Cambridge University Press.

Moss, S. (2012). On the Pragmatics of Counterfactuals. *Noûs*, 46(3), 561–586.

Nickerson, R. S. (2015). *Conditional Reasoning. The Unruly Syntactics, Semantics, Thematics, and Pragmatics of "If"*. Oxford: Oxford University Press.

Nolan, D. (1997). Impossible Worlds: Modest Approach. *Notre Dame Journal of Formal Logi*c, *38*(4), 535–572.

Nolan, D. (2003). Defending a Possible-Worlds Account of Indicative Conditionals. *Philosophical Studies*, *116*(3), 215–269.

Ramsey, F. P. (1931). General Propositions and Causality. In F. P. Ramsey, *The Foundations of Mathematics* (pp. 237–255). New York: Humanities.

Sanford D. H. (1989). *If P, then Q: Conditionals and the Foundations of Reasoning*. London: Routledge.

Sendłak, M. (2021). Counterpossibles, Story Prefix and Trivialism. *Synthese*, *199*, 7283–7301

Stalnaker, R. (1968). A Theory of Conditionals. In N. Rescher (Ed.), *Studies in Logical Theory*. Oxford: Blackwell.

Stalnaker, R. (1975). Indicative Conditionals. *Philosophia*, *5*(3), 269–286.

Todd, W. (1964). Counterfactual Conditionals and the Presuppositions of Induction. *Philosophy of Science*, *31*(2), 101–110.

Von Fintel, K. (2001). Counterfactuals in a Dynamic Context. In M. Kenstowicz (Ed.) *Ken Hale: A Life in Language*. Cambridge: MIT Press.

Williamson, T. (2016). Knowing by Imagining. In E. Kind, P. Kung (Eds.), *Knowledge Trough Imagining* (pp. 113–122). Oxford: OUP.

Williamson, T. (2018). Counterpossibles. *Topoi*, *37*(3), 357–368.

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Article

SAMUEL DICKSON *

AGAINST VACUISM

SUMMARY: This paper discusses the question of whether all counterfactuals with necessarily false antecedents (counterpossibles) are vacuously true. The orthodox view of counterpossibles (vacuism) answers that question in the affirmative. This paper explains vacuism before turning to examples from science that seem to require us to reason non-trivially using counterpossibles, and it seems that the counterpossibles used in such cases can be true or false. This is a threat to vacuism. It is then argued that the same kind of reasoning which produces non-trivial counterpossibles in scientific cases can be extended to the case of counterpossibles in mathematics. Ordinary counterfactual reasoning relies on rejecting background assumptions in order to assume the truth of the antecedent. A failure to perform this process in the counterpossible case is what leads one to vacuism and it is explained how this process produces non-vacuous; counterfactuals, scientific counterpossibles, and mathematical counterpossibles.

KEYWORDS: counterfactual, counterpossible, vacuism, non-vacuism, impossible worlds.

1. Introduction

Orthodoxy states that a counterfactual ($A > B$) is true when the nearest A-worlds are also B-worlds. For any counterfactual with an antecedent that logically implies a consequent, the counterfactual will come out true, regardless of the content of either part. If there are no A-worlds as described by the antecedent, then trivially *all* A-worlds are B-worlds, i.e., the counterfactual will come out as true (Stalnaker, 1968). Counterpossibles are a subset of counterfactuals that con-

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tain an impossible antecedent. For the purposes of this paper, we can assume that impossibility to be of the highest level. I will simply assume for now that this is the level of metaphysical necessity. So the counterpossibles I will generally be concerned with will be those with a metaphysically impossible antecedent, I will symbolise these as $A_i > B$. In virtue of being metaphysically impossible, it seems that there are no worlds at which A_i will be the case, so the orthodoxy tells us that any such counterpossible will come out as trivially true, regardless of subject matter. This theory is known as vacuism, one key proponent of vacuism is Williamson (2007; 2018). This paper mainly addresses his formulation of vacuism and his arguments for it, ultimately arguing that some counterpossibles are non-trivial.

Of course, no non-vacuists need to say that all counterpossibles are non-trivial, so many restrict the non-triviality thesis to specific domains. One place that it might be difficult to imagine the occurrence of non-trivial counterpossibles is in mathematical reasoning. Proofs by reductio seem to typically involve making impossible suppositions and then reasoning from them, ultimately proving that indeed the supposition is impossible and necessarily false. For these to work, it seems that all the statements in these proofs need to be true. This is exactly as the vacuist prescribes and so one might view this as a compelling argument to agree with vacuism. I disagree, and I think that the reasons we can give for believing in the non-triviality of other counterpossibles are extendable to the case of non-trivial countermathematicals. The basic argument I will offer is as follows: We have compelling reasons to think that there are non-trivial counterpossibles in the sciences, some scientific counterpossibles come out as false (and some true). This datum is significant enough to override the prescriptions of logical orthodoxy. Two things might be going on at this stage, either: we are implicitly using a non-standard semantics for counterfactuals in these cases, allowing them to come out with differing and non-vacuous truth values or; we are working within a standard semantics but still delivering this verdict, contra orthodoxy. It seems most likely that a vacuist would say such counterpossibles are true because there are no A_i worlds. It further seems that what might actually be going on in the cases of scientific counterpossibles is that we are genuinely considering an impossible world, and because the truth value of B is up for grabs at these A_i worlds, the truth value of the counterpossible as a whole can change. I will discuss how this is applicable to the case of countermathematicals.

This is the strategy I will be considering in this paper. We should genuinely consider the closest world at which any A_i is the case. Considering impossible worlds, on some minimal level allows us to deliver the verdict from science, it also shows us that vacuism is false. The unique contributions this paper aims to make lie in several places. As above, this paper aims to show that if we genuinely consider an impossible world/suppose that A_i, then different counterpossibles will have different truth values. This is illustrated in the cases of scientific counterpossibles discussed. This paper also aims to show that vacuism about counterpossibles in mathematics is a redundant thesis. Further contributions to the literature are made by distinguishing between two kinds of projects that one might undertake in counterfactual form. In the first case, one may wish to use counterfactual form to work out the truth value of the statement which forms the antecedent. The second case involves reasoning from the antecedent to potential consequents to see what would be the case, if the antecedent were true. Importantly for this second process, this is done regardless of the actual truth value of the antecedent, one has to genuinely consider it/suppose it to be true (Section 3.4 of the current paper). This distinction is a close companion of the distinction between a consensus and non-consensus context given by Yli-Vakkuri and Hawthorne (2020).[1] This paper aims to show that Williamson is engaged in the first kind of process, rather than the second kind of process. Even if all counterpossible statements in the first kind of process turn out to be true, it is not the case that counterpossibles used in the second process will, so vacuism is false. What Williamson (2007; 2018) does is to determine the truth value of a statement (A_i), which he does by embedding it as the antecedent in a counterfactual form. But this is different from genuinely considering what would be the case if A_i were true. Importantly, this genuine consideration is what Brogaard and Salerno (2016) are engaged in when responding to Williamson and this is the core reason that Brogaard and Salerno appear to be in disagreement with Williamson. They each think the other side is performing the same reasoning task and producing a different result, when in fact they are engaged in different enterprises. So this paper provides a methodological explanation of why the disagreement between vacuists and non-vacuists has arisen. It is also worth noting that, in the literature on counterpossibles, it is often the case that non-vacuists will provide examples of counterpossibles that are non-vacuous (e.g., Jenny, 2018), but not necessarily provide a general overarching explanation for their non-vacuity. They say that the counterpossibles in question are non-vacuous, but not always why. This paper aims to start providing an answer to that question by pointing to the use of non-vacuous counterpossibles in scientific explanations, and showing how the mathematical cases mirror this.

As a final prelude before starting the discussion, it will be worth clarifying some assumptions at play. It is worth stating up front that I am implicitly assuming some variation of a Lewisian conception of worlds,[2] which includes an account of impossible worlds. Although I have some reservations about the specific account, Yagisawa's (2010) extended modal realism is an interesting take on impossible worlds and the general spirit of that account can be kept in mind when impossible worlds are mentioned in this paper. Given an account of both possible and impossible worlds, I think it is very plausible that one can maintain the standard semantics of Lewis-Stalnaker, because if there are impossible worlds, then we can assess counterpossibles on the basis of the closest one.

[1] This also seems close to the suppositional procedure that Williamson describes in Suppose and Tell (2020). However, as will be argued for later on, I think Williamson fails to properly engage in the suppositional procedure, and that is why he believes the counterpossibles to all be true.

[2] Along with the associated semantics.

However, it is also worth noting that is obviously not an inherent commitment of non-vacuism, one can be a non-vacuist without believing in this specific conception of impossible worlds. One need not even accept impossible worlds at all, perhaps one way to do this is to alter the standard Lewis-Stalnaker semantics instead.[3] Although one might say that an appeal to either a different ontology of possible worlds or a different semantics is problematic, it is worth noting that the only reason that Williamson thinks he can achieve a vacuist result is by assuming a specific semantic account/a specific conception of worlds, so if this is a problem for non-vacuists, it is equally a problem for vacuists. One way to read the following arguments about scientific and mathematical practice and the treatment of counterpossibles is that they provide reasons to think that experts in those disciplines make assumptions close to the ones described above, and that provides us a reason to make them too, rather than the ones that vacuists make. With these clarifications in place, we are in a position to begin considering counterpossibles.

2. Counterpossibles in Science

2.1. Tan's Cases

There are a plethora of examples of counterpossible pairs that intuition tells us have different truth values. But intuition only takes us so far, the vacuist can simply say this is the appearance of the distinct truth values, but the logical form tells us we are actually mistaken. This response by the vacuist will not work in the scientific case. If good scientific practice leads us to assign some counterpossibles as being false, we need to account for this. The results from science outweigh philosophical/logical inclinations we may have. Compare this with how developments in quantum mechanics have led some to alternative quantum logics to account for the discrepancies (e.g., Putnam, 1969), of course such usages are controversial and by no means the orthodox, but this shows that it is not universally agreed that classical logic always has the correct verdict. The usage of counterpossible reasoning in the sciences is documented by a number of people (McLoone, 2020; Wilson, 2021). One such discussion takes place in Tan (2019), in which he presents examples of the use of non-trivial counterpossibles in science. Not only are there multiple examples of counterpossibles used in science, but they are used in different ways and for different purposes. Tan (2019) focusses on their use in: scientific explanation; idealised scientific models; and in reasoning about superseded scientific theories. In each of these cases, he offers an archetypal example of a counterpossible and discusses why viewing it as counterpossible and as non-trivial is the correct verdict. In the case of scientific explanation, the counterpossible offered is:

[3] Another way to do this would be to adopt some appropriate form of non-classical logic. Whilst I do not wish to rule out this route, it will not be discussed in this paper.

(A) "If diamond had not been covalently bonded, then it would have been a better electrical conductor" (Tan, 2019, p. 40).

Tan claims that this is a scientific explanation of the fact that diamond cannot conduct electricity whereas solid carbon in some other forms can. The reason the covalent bonding explains this fact is because covalent bonds do not leave free electrons, as they "use up" all the electrons forming the strong bond. In other substances, free electrons allow for electrical conductivity (Tan, 2019, p. 40). The property of poor conductivity that diamond has is brought about as a result of these bonds, and so the microphysical structure. This counterfactual then provides an explanation in virtue of highlighting that dependence relation. But one might wonder if this is indeed a counterpossible; one may wonder whether diamond could have been otherwise bonded, in which case this would be a mere straightforward counterfactual. One can approach this in two ways, we might consider whether something is called diamond in virtue of its microphysical structure or in virtue of its theoretical role in science (Tan, 2019). Going the first route, one can easily see that this is a counterpossible, because if something is only diamond in virtue of its microphysical structure, then something which had a different microphysical structure would not be diamond. As a matter of metaphysical necessity, diamond has the structure that it does. So it is metaphysically impossible for diamond to be differently bonded.

Going the second way, one may think that we define diamond by its theoretical role, the diamond-stuff is the stuff that does *x*, *y* and *z*. But the reason diamond is distinguished from other substances, and the reason it does the things it does, is because of its microphysical structure. In other words, nothing else could do the things diamond does without its microphysical structure. Nothing could fill the diamond role without actually being diamond. So again, it is metaphysically impossible that diamond could have been differently bonded than it in fact is. So it seems then, that statement A above is a counterpossible. Tan (2019) goes further than this, he insists that this is also a counterpossible which is true, and non-vacuously so. This is because it describes an empirical fact, that the poor conductivity of diamond physically depends on its microphysical structure. So science relies on non-vacuous counterpossibles in scientific explanation (2019, p. 42). One can easily see how this is not an isolated case because many scientific explanations of why substances have the properties they do will rely on a similar explanatory structure.

As stated, Tan also thinks that we need to make use of substantially true counterpossibles when reasoning about superseded scientific theories. Sometimes, we need to reason about scientific theories using counterfactuals; "If Jupiter were a point mass then…" and "If classical mechanics had been true…" are examples of each of these (Tan, 2019, p. 48). As Tan points out, we might counterfactually reason about a false theory to describe its empirical content, e.g., "had the geocentric Ptolemaic system been correct, celestial spheres would be unobservable entities". Counterfactual reasoning is also used in order to explain the falseness of a false scientific theory. Tan considers a straightforward example of this concerning Bohr's theory of the atom (Tan, 2019, p. 48):

(B_1) If Bohr's theory of the atom had been true, then an electron's angular momentum, *L*, in the ground state would have been observed at $L = h$ (the reduced Planck constant).

(B_2) It is not the case that the electron's angular momentum, *L*, in the ground state is observed at $L = h$.

(B_3) Therefore, Bohr's theory of the atom is false.

Bohr's theory of the atom predicts/requires that the angular momentum of an electron is observed in the above way, that is to say that (B_1) is correct. Given that that is a result of the theory, if the theory were correct then that would be the case. Repeated experimentation and observation has shown that the angular momentum of an electron is actually zero in the ground state, i.e., (B_2) is true. Given that both (B_1) and (B_2) are true, it then simply follows that (B_3) is true. This is a substantial result, and clearly (B_1) is true more than merely trivially. As Tan puts it: "in order for this commonplace pattern of reasoning to be epistemically fruitful, theory-evaluating conditionals must describe genuine relations of counterfactual dependence and implication. They must, in other words, be non-vacuously true" (Tan, 2019, p. 49).

This seems to be correct, the above essentially takes the form of "if *that* were right, we would see *this*. We do not see *this*, so *that* must be wrong". We want such arguments to produce truth that is not merely trivial, because the process Tan talks about seems like an example of good scientific reasoning. There are many examples of this process being used in the sciences for all manner of theories. As a method of theory falsification, it is a good one, and we need it to produce substantive, non-trivial results. Now one may be willing to accept this but unwilling to extend it to the counterpossible case, because of a commitment to vacuous counterpossibles. The problem here is that (B_1) is already a counterpossible. This archetype of non-vacuous scientific reasoning turns out to involve counterpossible reasoning. If one wishes to trivialise all counterpossibles then one is going to have to trivialise a lot of scientific reasoning, and this seems an unattractive feature of any account. The reason that (B_1) is a counterpossible is that Bohr's theory of the atom is an inconsistent theory. It rests on both classical and quantum assumptions, therefore some aspects of the theory represent orbiting electrons as radiating energy as they move about; other aspects of the theory represent electrons as non-radiative (Tan, 2019, p. 49). In other words, the theory as a whole contains a contradiction, as it represents electrons both as radiating energy and as not radiating energy. (B_1) does not merely refer to one aspect of Bohr's theory, it refers to the theory as a whole, and the theory as a whole contains this contradiction. So it is simply logically impossible that Bohr's theory of the atom be true, it is impossible that Bohr atoms could exist. (B_1) then, is a counterpossible. But we have already established that (B_1) is non-vacuously true. A potential response from vacuists could be that we can maintain vacuism because we accept that (B_1) is true (vacuously) and also accept that (B_1*) is true:

(B_1*) If Bohr's theory of the atom had been true, then an electron's angular momentum, *L*, in the ground state would *not* have been observed at $L = h$ (the reduced Planck constant).

(B_1*) negates the consequent of (B_1), but as it is a counterpossible, is also true (vacuously so). The vacuist might respond that the reason we appeal to (B_1) rather than (B_1*) is because the former has proved useful for scientific progress and prediction due to the way the world happens to be, whilst the latter has not. The problem I see with this response is that I do not think particle physicists would accept that (B_1) and (B_1*) are equally true. It seems much more likely that physicists would judge (B_1) to be true, but (B_1*) to be false. Now the vacuist may point out that orthodox philosophical practice leads us to conclude that both counterpossibles are vacuously true. But there is nothing to stop the particle physicist from pointing out that scientific practice leads us to conclude that one is true, and the other false. In short, the scientist need not be persuaded by what the vacuist has to say. Furthermore, if we are to base our judgments on the views of either, it seems we should base them on the views of the scientists regarding these scientific matters, rather than what the philosopher thinks about the truth/falsity of these statements.

Another place that Tan (2019) alleges science makes use of counterpossibles is in reasoning with idealised models. Science often treats planets as points for the purposes of performing calculations on their gravitational effect. Sometimes scientists also treat planes as if they are frictionless and liquids as if they are continuous. The use of such idealised modelling is prevalent throughout science, and once again arguably essential. For example, the sheer complexity of modelling a liquid as a series of discrete but bonded particles makes performing such calculations so difficult as to be unproductive, if not downright impossible. So scientists do tend to model things *as if they were these* idealised things. Tan (2019) alleges that these idealised things could not exist and could not fill the role of the substance being tested/investigated. For example, a continuous incompressible liquid could not do the things that water does, it could not be water. Yet we model water as if it were such an idealisation. Tan's claim is that we are modelling an impossible situation. Furthermore, reasonings based on such impossibilities constitute counterpossibles, e.g., "had water been a continuous incompressible medium…" (2019, p. 46). Such modelling is useful because the behaviour of water as it actually is closely approximates that of a continuous incompressible medium. The antecedent of this counterfactual model, i.e., "had water been a continuous, incompressible medium…" is metaphysically impossible. This makes the statement, as a whole, a counterpossible. Furthermore, it is a non-trivially true counterpossible.

We can explain why this statement is a counterpossible in similar ways to the diamond case. It is held that necessarily, water is identical to H_2O. As such, water has to be built up out of H_2O, and nothing that is made of anything else can be water. H_2O is not a continuous, incompressible medium, it is a series of bonded but discrete particles. So if something was such a strange medium, it would not be H_2O (and so not water). It would be metaphysically impossible for water to be a continuous, incompressible medium. But maybe people are not convinced here, again, perhaps they wish to define water by its theoretical role, rather than its chemical composition. Tan (2019, p. 46) thinks that even this view would lead to the statement in question being a counterpossible. One might allege that perhaps some continuous, incompressible medium can fulfil the role of water by acting exactly as actual water does. The problem is that this simply cannot be the case, a continuous, incompressible medium cannot fulfil the role of water. For example, a key property of water is that it is a solvent for particulate solids. No continuous, incompressible medium could ever act as a solvent for particulate solids, so no continuous, incompressible medium could ever fulfil the causal role of water (Tan, 2019, p. 46). Again, however we are defining water, it is metaphysically impossible that it be a continuous, incompressible medium. Yet we model it as such, so such models constitute counterpossibles.

One may be willing to accept this but deny that this counterpossible is non-vacuously true (or false). Tan's answer to this is to point to scientific practice and how things are actually done (and indeed how they have to be done). He alleges that such practices require us to treat these counterpossibles as non-trivially true. Tan uses the example of two competing models about the behaviour of water, M_1 and M_2. They both represent water as an idealised continuous fluid but they differ with respect to the viscosity they ascribe to water (2019, p. 47). To test these models, scientists will see how close the behaviour of water is to each model. Let us imagine they discover the predictions of one theory, M_1 to be very close to the behaviour of water, whilst the predictions of M_2 are further off. Scientists would rightly judge M_1 to be a true (or approximately so) theory, whilst M_2 would be false. Furthermore, they would take the following counterpossible to be false:

(C_1) "If water were a continuous, incompressible medium, then it would behave as M_2 predicts"

whilst taking this one to be true:

(C_2) "If water were a continuous, incompressible medium, then it would behave as M_1 predicts" (Tan, 2019, p. 47).

As we have already established, both are counterpossibles, and yet they have their truth values non-trivially. Orthodoxy might dictate that both of these are vacuous, but this does not constitute an argument for that being the case. Furthermore, the fact that it seems a worthwhile endeavour to reason using such counterpossibles is in fact evidence against the orthodoxy. If scientists were unable to reason so, then a large swath of scientific practice would disappear. Scientists need to use models like this and do so fruitfully, this would not be possible from vacuous counterpossibles, so we need to hold them to be non-trivial.

I think vacuists will struggle to respond to such cases from science. Scientific practice seems to require us to treat counterpossibles non-trivially, and this is important. The vacuist may have to say that scientists are simply mistaken, but this is unattractive as a position. Nor is it a position that scientists are likely to accept. If our logic/semantics conflicts with successful scientific practice then this seems to indicate a flaw in the logic/semantics rather than the scientific practice. Given this, non-vacuism may seem preferable. It will be helpful to consider one line of response the vacuist might make which I think fails. A vacuist could easily respond that indeed scientific practice does require us to treat some counterpossibles as non-trivial, but that this is not because such counterpossibles are non-trivial. Instead, perhaps what matters is that some scientific counterpossibles are assertable and some not, these are the ones we treat as non-trivial.

2.2. Assertability

A vacuist might say that the counterpossibles I want to describe as false are in fact merely not assertable (as discussed by Grice, 1975) and the ones I want to describe as non-vacuously true are assertable. This can be the case whilst all of them are true, and so I have not shown the vacuist thesis to be false, I have merely shown that some counterpossibles are assertable, and some are not. Perhaps, the class of "true" counterpossibles are assertable because they point to some underlying non-counterpossible truth, whereas the "false" counterpossibles fail to do this. For example, take the following pair of counterpossibles (Emery, Hill, 2017, p. 136):

(1a) If Obama had had different parents, he would have had different DNA.

(1b) If Obama had had different parents, he would have been two inches tall.

(1a) is assertable because it points to the underlying fact "(1c) Obama's parents were the cause of his having the DNA that he has" (Emery, Hill, 2017, p. 138). Whereas (1b) does not. Because they fail to do this, such counterpossibles are not assertable, and we mistake this intuition and say that they are false (Emery, Hill, 2017, pp. 137–138). However, as the orthodox view shows us, such intuition is mistaken, as all counterpossibles are true. The assertability of a statement, *s*, such as (1a) and the unassertability of its converse *s**, such as (1b), does not imply that *s* is true and *s** is false. The vacuist can then account for the views of non-vacuists whilst maintaining their theory.

This is an interesting point, but I do not think it threatens my view. Firstly, if it is the case that, for a given conflicting pair of counterfactuals, the assertability of one and the unassertability of the other does not imply that one is false, then it is also the case that it does not imply that they are both true. As Sendłak (2021) argues, the same pattern of assertability can be found in non-counterpossible counterfactuals, and whilst failing to imply that one is false, it also does not mean that both become true, for example as Sendłak says:

> [T]he assertion of "If Christopher Columbus had reached the place he was planning to reach in 1492, he would have arrived in India" can be explained by the fact that this allows one to indirectly express a more substantial proposition that is related to the asserted proposition in subject matter, e.g., "Christopher Columbus was planning to reach India". (2021, p. 11)

Whereas the converse "If Christopher Columbus had reached the place he was planning to reach in 1492, he would not have arrived in India", should intuitively be false, but under the Emery and Hill analysis, the truth value of the first sentence should not affect the truth value of the second, and so we could also view it as true. But crucially we can explain that the reason we intuitively think it is false is due to its unassertability. Sendłak claims that if we view this as problematic in the counterpossibles case, it is equally problematic in the counterfactual case, and that one could then hold a vacuist view of counterfactuals. Given the intuitive falsity of vacuism about counterfactuals, this is obviously a problem for a vacuist account that would endorse this (Sendłak, 2021, p. 11). Whilst it is true that a statement can fail to be assertable (for various reasons) without failing to be false, it does not mean that each and every statement which fails to be assertable also fails to be false. Emery and Hill (2017) try to introduce a gap between the unassertability of something and its falsity, the problem in the way they do this is that it creates a total disconnect between assertability/unassertability and the truth of a statement, in doing so they miss the target they aim for.

As noted, the result that science seems to rely on non-trivial counterpossibles is significant. Moreover, it is arguably a result we should favour over the traditional semantics. If scientists need to treat counterpossibles as non-trivial, then our accounts of counterpossibles need to treat them as non-trivial. One way in which a defender of vacuism might respond is to say that scientists do not need to treat counterpossibles as non-trivial, instead treating them as trivial but assertable/unassertable. This would not work though, the kind of arguments used for this could also be used to show that ordinary counterfactuals are trivial. This is clearly false, so something must be faulty with the argumentation. This way of saying that counterpossibles are merely assertable/unassertable will not work.

At this stage, the most we can have shown is that at least some counterpossibles are non-trivial, plausibly a large class of scientific ones. This of course does not show that all counterpossibles are non-trivial. As we noted at the start, on the face of it there might seem to be a difficulty with non-vacuous countermathematicals. Given that we need all counterpossibles in proofs by reductio to be true, the vacuist seems to be in a strong position. I think we can extend the spirit of why scientific counterpossibles are non-trivial to the case of countermathematicals and show that there are also non-trivial examples. First, it will be worth going over Williamson's (2018) discussion of why countermathematicals should be vacuous, as it will highlight some important points.

3. Counterpossibles in Mathematics

3.1. Williamson's Case

Williamson (2018) discusses the use of counterpossibles in mathematical proofs using reductio ad absurdum. As a hallmark example of this, he uses the proof that there is no largest prime number, known as Euclid's theorem. Williamson stresses that one does not necessarily need to phrase mathematical proofs in terms of counterfactual conditionals, but that it is a legitimate and natural way of doing so. So regardless of particular views on counterpossibles, all parties need an explanation of why this reasoning is legitimate and works. Williamson borrows the example from Lewis (1973, p. 25):

(L) If there were a largest prime, p, $p! + 1$ would be prime.

(M) If there were a largest prime, p, $p! + 1$ would be composite.

Williamson (2018, p. 363) helpfully summarises this proof: of (L) he explains that it holds because "if p were the largest prime, $p!$ would be divisible by all primes (since it is divisible by all natural numbers from 1 to p), so $p! + 1$ would be divisible by none" (2018, p. 363). Of (M) he points out that it holds because "$p! + 1$ is larger than p, and so would be composite if p were the largest prime" (Williamson, 2018, p. 363). Given that both these conditionals have the same antecedent, we are entitled to conjoin their consequents, resulting in:

(N) If there were a largest prime p, $p! + 1$ would be both prime and composite.

Given that the consequent of this counterfactual is a contradiction, we can deny the antecedent, and so say that in fact there is no largest prime. Quite obviously these are counterpossibles as well, because there cannot be a largest prime, that is a mathematical impossibility. Williamson and other vacuists, along with non-vacuists, will accept this as a good mathematical proof. In other words, everyone should accept all of (L)–(N) as true. Williamson's strategy is then to offer another proof by contradiction, using vacuous counterpossibles, which he says vacuists can accept easily, but that non-vacuists cannot accept, and cannot reject without rejecting Euclid's theorem. If non-vacuists deny the truth of the premises in Williamson's proof, he alleges they must also deny the truth of the premises in Euclid's theorem. Since rejecting such a proof would be unacceptable, we have a strong reason to doubt non-vacuism; so Williamson's argument goes. Before explaining why I do not think this argument works, I will spell out Williamson's second proof.

Williamson asks us to consider someone who answered "11" to "What is 5 + 7?" but who mistakenly believes that they answered "13", and utters the following counterpossibles, for the non-vacuist, (O) is false, whilst (P) is true (2007, p. 172):

(O) If 5 + 7 were 13, I would have got that sum right.

(P) If 5 + 7 were 13, I would have got that sum wrong.

Williamson is not persuaded by the initial intuitiveness of such examples:

> [T]hey tend to fall apart when thought through. For example, if 5 + 7 were 13 then 5 + 6 would be 12, and so (by another eleven steps) 0 would be 1, so if the number of right answers I gave were 0, the number of right answers I gave would be 1. (2007, p. 172)

If the number of right answers the person gives is 0, i.e., they give a wrong answer, then the number of right answers they give is 1, i.e., they get the sum right. So both counterpossibles are going to turn out to be true. Williamson then asserts that this is a result that the vacuist can get and accept, but that the non-vacuist cannot. He claims this points in favour of vacuism about counterpossibles. However, there is room for debate here. In particular, Brogaard and Salerno develop a series of objections against Williamson's reasoning.

3.2. Brogaard and Salerno's Objection

Brogaard and Salerno (2013) analyse Williamson's argument a bit more in depth and draw out the extra steps Williamson himself alludes to. The conclusion Williamson draws is that "if the number of right answers I gave were 0, the number of right answers I gave would be 1", hence, both (O) and (P) are true. The steps that Williamson abbreviates will be something akin to, if not exactly the following (Brogaard, Salerno, 2013, p. 649):

(i) If 5 + 7 were 13, then 5 + 6 would be 12.

(ii) If 5 + 7 were 13, then 5 + 5 would be 11.

…

(xi) If 5 + 7 were 13, then 5 + – 4 would be 2.

(xii) If 5 + 7 were 13, then 5 + – 5 would be 1.

It seems to be that what Williamson's argument is, at this point, is that worlds in which 5 + – 5 = 1 are also worlds in which 0 = 1, because we can substitute 5 + – 5 for 0. So we can conclude that:

(xiii) If 5 + 7 were 13, then 0 would be 1.

And so we get to Williamson's (2007, p. 172) conclusion that "if the number of right answers I gave were 0, the number of right answers I gave would be 1", with (O) and (P) both being true. Brogaard and Salerno go on to object that we can reject Williamson's proof here because he does not do a good enough job in establishing that the closest impossible world in which 5 + 7 = 13 is also one in which 5 + 6 = 12 (2013, p. 650). At this stage, we can return to Williamson's (2018) argument against non-vacuism.

The charge is that if non-vacuists reject Williamson's proof on the grounds that we have not established that the described world is the closest impossible world, then they must also reject Euclid's theorem for the same reason. Mathematicians will not concern themselves with the relative closeness of impossible worlds when producing proofs by contradiction, they will just produce the proof. So there is no evidence that the closest impossible world in which there is a largest prime, *p*, is also a world in which *p*! + 1 is both prime and composite (Williamson, 2018, p. 363). Non-vacuists will then be compelled to either reject Euclid's theorem, or to find a way of showing that the closest impossible world in the prime number case is indeed the world that Euclid's theorem describes. However, there is of course no guarantee that the same process cannot be performed for Williamson's proof, which would seem to tell against the non-vacuist. Essentially then, we should be viewing both counterpossibles in both cases as true, this is exactly as the vacuist describes and expects, but not as the non-vacuist does (Williamson, 2018, pp. 363–364). Having seen Williamson's argument we are in a position to respond to it. I think, at this stage, it will be worth making some clarifications about vacuism, and what Williamson has established so far, and also to build upon Brogaard and Salerno's objection, because whilst it might not work in its current form, I think there is an important idea contained within it.

Williamson claims that the counterpossibles used in Euclid's theorem and in his own proof are all true, because they follow from mathematical reasoning. The vacuist can obviously account for this, but the non-vacuist cannot, so Williamson claims. Perhaps the non-vacuist intuition that, for example, (O) and (P) have different truth values stems from some commitment that for any pair of counterfactuals that have contradictory consequents, but the same antecedent, at least one must be false. Williamson will claim that this failure to deliver the verdict of mathematics is a significant drawback of the non-vacuist account, and so we should reject such an account. I think non-vacuists can respond to this though. Not only has Williamson failed to successfully establish vacuism, I do not think that these mathematical proofs even constitute an argument for it.

3.3. The Problem With Vacuism

One could say that non-vacuists do not have to reject Williamson's proof. Certainly, Brogaard and Salerno did so under the banner of non-vacuism, but this is not an inherent commitment of that theory. There is nothing inherent in non-vacuism that says one cannot accept Williamson's proof. Perhaps Williamson has shown that all those counterpossibles are true, but that does not mean he has shown that vacuism is true, or even that all counterpossibles are true. Vacuism is essentially the thesis that all counterpossibles are vacuously true, because their antecedents are necessarily false.[4] The truth of the counterpossibles comes from this fact, this is what makes the counterpossible true. The problem is that vacuism plays no role in making (L), (M) or (N) true in Euclid's theorem. As Williamson himself says, they are true because they are mathematical results; "[(L)–(N)] should be true, for they are soundly based on valid mathematical reasoning" (2018, p. 363). But this is independent of vacuism. Williamson correctly points out that a semantic theory needs to produce this result, and indeed vacuism does, but for one it is unclear that it does so for the correct reasons, and two, it is not the only semantic theory that does this. The truth of (L)–(N) is a mathematical result, they are true for reasons stronger than the mere impossibility of the antecedent. Compare this with:

(L*) "If there were a largest prime *p*, *p*! + 1 would be a set", or

(L**) "If there were a largest prime *p*, *p*! + 1 would be an infinite set".

I think mathematicians would want to reject these conclusions, they would want to say that these statements were false, as would non-vacuists. They would be false because they would be based on faulty mathematical reasoning. However, on Williamson's account, they would come out as true. Consider a world in which mathematical practice was systematically wrong. For whatever reason, mathematicians just get the wrong verdict when talking about these matters. In such a world, clearly some counterpossibles would be described as false by the mathematicians, but they would all be described as true by the vacuist. The result from vacuism and the result from mathematical practice are distinct results. I think this in itself constitutes a criticism of vacuism. We have already discussed cases in science that seem to require non-trivially true/false counterpossibles, so it seems vacuism about all counterpossibles might be false. But restricting vacuism to mathematical counterpossibles is a redundant thesis, this amounts to a claim that all the mathematically proven statements are true. Or, if mathematical practice told us that a particular counterpossible was false, it would amount to disagreement with mathematical practice. This second alternative is exactly what the vacuist charges the non-vacuist with as a significant problem, and yet it seems they might be vulnerable to exactly the same point. But the point against Williamson and the vacuists is not merely that his account produces the wrong results in certain cases, that would merely be a reframing of the intuitive arguments for non-vacuity. Instead, the point is that in the mathematical cases he would appeal to, although he gets the right result, the result is obtained regard-

[4] Non-vacuism of course being the thesis that there are at least some non-trivial counterpossibles.

less of his theory. We can see this by the fact that non-vacuists accept the result that both statements are true in the case of Euclid's theorem, and they do so on non-vacuist grounds; because it is a mathematical result. Williamson's mistake comes from the fact that he assumes that, to take the counterpossibles he makes in his proof as being true, the non-vacuist would have to subscribe to some form of vacuism; but this is not the case. One can take (L)–(N) to be true without being a vacuist,[5] and that is so because, as Williamson points out, they follow from mathematical reasoning. Our intuitions led us to think that (O) and (P) had different truth values, but mathematical reasoning showed us this was wrong. That is something the non-vacuist can accept, just because non-vacuism is committed to some counterpossibles being non-trivial, it does not mean that on each occasion that our intuition points to counterpossibles having different truth values, we are right. Importantly again, the mathematical counterpossibles we have discussed are not even trivially true. They follow from mathematical reasoning so they are substantially true.

We have seen how Williamson's proof works and how the non-vacuist can equally accept this result. Williamson's proof does seem to fall out of standard mathematical definitions of addition, the successor principle, etc. But another point to be considered is whether or not Williamson has genuinely evaluated the truth value of the counterpossible in the way it should be. One important point to discuss is the Baron, Colyvan and Ripley (2017) discussion that Williamson's proof fails to consider the closest counterpossible scenario. But first it will help to consider an important distinction that I think is very relevant to the current topic, the distinction between genuinely conceiving of a distinct world, and considering a conjecture at the actual world.

3.4. How to Genuinely Consider a Distinct World

I think Brogaard and Salerno (2016) have captured something with their objection. They charge Williamson with not conceiving of the closest possible world. Williamson says that rejecting his proof on these grounds would mean we also have to reject any mathematical proof by contradiction, such as Euclid's theorem. This is clearly unattractive, and so we should not reject his account. But I think that this objection has targeted something important, albeit in the wrong way. Williamson's proof does not work by describing the closest world (in which the conjecture is true) to the actual world, but this is because his proof does not consider a distinct world at all. What Williamson has done is show that the actual world cannot be a particular way, given what we already know. This is a point worth spelling out in some detail.

Let us consider the two different kinds of process we might engage in using counterfactuals that were mentioned in the introduction. In the first case, the

[5] Indeed, it seems that one can understand and reach this result, without any view on the vacuity/non-vacuity of counterpossibles.

truth value of a statement/hypothesis might be unknown, and so we want to find out/demonstrate whether it is true or false. To do this, we use the hypothesis to derive a prediction and make a counterfactual using the hypothesis as the antecedent and the prediction as the consequent. If the prediction turns out not to be the case, we can use this to show that the antecedent was false. This is what we are doing in the example of Bohr's theory and in Williamson's proof. We say if one thing were the case, a second thing would also be the case, as the second is not the case, we can say that neither is the first. If 5 + 6 were 13, then 0 would be 1, 0 is not 1, so 5 + 6 is not 13. Now as it happens, in both these cases, the antecedents turn out to be necessarily false, and so the counterfactuals involving them are actually counterpossibles. The vacuist says that as counterpossibles are trivially true, these particular ones are trivially true. However, these particular counterpossibles are useful. The counterpossibles that non-vacuists wish to call true, (B_1 and xiii) contain consequents that contradict our experience, as such these are the ones which can actually be used to show the antecedent to be false. This is the process one might engage in to show that the antecedent of a counterfactual is false, and this is the process that Williamson is engaging in. However, there are situations where we already know the truth value of the antecedent, and these are the cases I want to focus on.

There may be cases when we know that a statement is false, perhaps even necessarily false, but we want, for whatever reason, to explore what would be the case if in fact it were true.[6] This is what we are doing in the case of modelling water as a continuous medium and in the Brogaard and Salerno example. In these cases, we know that the antecedent is false, we know that water is not an incompressible, continuous medium, and we know that 5 + 6 is not 13, but we want to find out what would be the case if they were. In order to find out what would be the case if they were true, we have to assume them to be true. To do that, we need to sacrifice some assumptions to avoid contradictions, e.g., that water is not a continuous medium and that 5 + 6 is not 13. Doing this would prevent us running into contradictions and so the counterpossible would not be trivial, because we could produce a false counterpossible by making a false statement about what would be the case if the impossible antecedent were the case. It is worth pointing out that we already make the distinction between these kinds of projects in the case of ordinary counterfactuals. Let us take the case of a crime scene investigation; in conjecturing how the murder victim was killed, the detective will make hypotheses. Perhaps one of these hypotheses is that the victim was shot. The detective may then form a counterfactual of the form "if it were the case that the victim was shot, there would be a gunshot wound on the body". If no gunshot wound is found, the detective can conclude that the ante-

[6]A similar idea to what follows occurs in Sendłak's (2021, pp. 16–18). However, this idea presents an important critique of vacuism concerning counterpossibles and so I think it warrants more attention and exploration than it has been given elsewhere.

cedent was false. In such a case, counterfactual reasoning has been used to discover that something is false.

Alternatively, sometimes we know that a statement is false, but we want to work out what would be the case were it true. If you cycle to work and your tyre bursts, resulting in you being late to work, you can usefully say “if I had driven to work, I would not have been late”. We know the antecedent is false, but we assume it to be true, and reject assumptions like you actually having ridden your bike in order to make non-trivial statements. If we did not reject assumptions, we would simply run into contradictions and end up proving that you had in fact cycled to work, but this is not what we wanted to do. This distinction between kinds of reasoning is present in the case of ordinary counterfactuals and it is not clear why it should not be present in the case of counterpossibles. With this distinction more clearly in mind, we can assess Williamson’s account of counterpossibles. I think we can diagnose why Williamson thinks he has got the result he does, whilst also explaining Brogaard and Salerno’s objection. Put simply, Williamson is engaged in the first kind of reasoning process mentioned above, whilst Brogaard and Salerno are engaged in the second.

Williamson’s proof is simply a proof that $5 + 7 \neq 13$. That is a perfectly legitimate thing to do and might be useful in some circumstances. But the reason that proof works, is the same reason the Euclid proof works. It works because we hold fixed everything we know about the world (in this case mathematics), and then show that given that, a particular fact could not be the case. In Euclid’s proof, we hold fixed facts about prime numbers, where in the number sequence they tend to appear for example. We then want to show that the assumption that there is a largest prime number is inconsistent with this. In doing this, we have not considered a different world, we have not moved from our world. Because we are showing that something cannot be the case, *at our world*. In Williamson’s proof, he has perhaps held fixed facts about addition, the successor principle, etc. and then shown that given these things, $5 + 7 \neq 13$. But note, this is not to consider a world in which 5 + 7 is 13. Because if we are considering a world in which $5 + 7 = 13$, this cannot be a world in which it is also the case that $5 + 7 \neq 13$. Williamson has not considered a different world, he has considered the actual world and shown that a certain statement is false here. Now, all the statements Williamson invokes might be true, but once again, they would be true non-vacuously, because they would be mathematical results. But it is not clear that he is genuinely considering a counterpossible.

Williamson (2020, p. 18) describes a process he calls the Suppositional Procedure (SP). In order to assess the truth of a conditional if *A* then *C*, one has to suppose that *A* and then judge whether, on the basis of that, it is also the case that *C*. Importantly, this simple form of the SP makes no mention of the possibility of *A* or *C*, simply that one must suppose *A*. One intuitive claim about supposing is that we have to suspend our disbelief in some way, perhaps just as in the case of make-believe games. Leng (2010) talks about make-believe in mathematics and describes the process as representing real objects in some way. Specifically it is to

> [i]magine of real objects that they are other than they really are. It is clear in these cases that we are sometimes being required to imagine something *false* concerning the nature of such objects: we know that the tree stumps are not *really* bears; that the fluids are not *really* continuous. (Leng, 2010, p. 159, author's emphasis)

If we know *A* to be false, but want to suppose it for some purpose, we have to reject other facts which would rule *A* out. The move I wish to make should be clear now, in Williamson's proof above, he has simply failed to suppose[7] that 5 + 7 = 13. Let us consider a more in depth spelling out of a true suppositional process.

Take any proposition, *P*, if one is to consider a world at which it is the case that *P*, then the world considered must also be a world in which ~~*P*. Now this is not to say that there cannot be worlds which contain contradictions. If we are considering a world in which it is raining and not raining (same place, same time), it seems like we are considering a world in which *P* and ~*P* (neglecting to include ~~*P*). But this misses the mark a little bit. We are considering a world in which it is the case that it is raining and it is not raining. This is a proposition, *Q*. If we need to consider that world, then we also need to be sure that it is a world at which ~{~[it is raining and it is not raining]}, i.e., that it is also a world at which ~~*Q*. Williamson fails to consider a world at which 5 + 7 = 13, because he does not ensure that it is also a world at which it is the case that ~{~[5 + 7 = 13]}. And holding fixed the background mathematical facts, just as in the Euclid case, is key to the proof working, because the proof aims at showing that *at the actual world*, something is not the case. It is worth noting as well, that is not some method peculiar to counterpossibles. This is exactly the process we need to engage in for ordinary counterfactual scenarios.

Let us take the straightforward counterfactual "If Julius Caesar were alive today then…". We have a number of assumptions that we are committed to at this world, the average lifespan of a human being currently sits at around 81 in the UK. Perhaps given this, we also assume that anyone who was alive at the time of Julius Caesar is now dead, including Caesar himself, i.e., we assume that ~[Julius Caesar is alive today]. In order to genuinely consider a world at which it is the case that Caesar is alive today, we need to reject this implicit assumption for the purposes of conceiving. We need to explicitly make sure it is a world at which ~{~[Julius Caesar is alive today]}. If we do not do this, then we will of course run into inconsistencies, and potentially end up proving that our conjecture (that Caesar is alive) is incorrect. But this is not to genuinely conceive of a distinct world, this is a different process. If we were to consider the closest world in which 5 + 7 = 13, then we are going to have to jettison some mathematical assumptions. In doing so, it is not clear that all of Williamson's statements would follow mathematically, and so be true. In fact, it actually seems more likely that Williamson's proof will not go through, some statements will come out false. In just the same way as if we genuinely considered a world at which

[7] A more in depth discussion of why I assert this takes place in Section 4.3.

there was a largest prime, likely Euclid's theorem would not work. But this should not be surprising, a world with a largest prime is a world where Euclid's theorem is false.[8] This does not threaten mathematical practice, because this is not the aim of mathematical practice.

There are of course limits to how far this process can go, both in terms of unavoidable contradictions and in terms of the considered scenario being so distant from our own as to be irrelevant. But such things can be assessed on a case by case basis, Baron et al. (2017) propose a method in this style for "chasing out" contradictions from the immediately relevant vicinity of the counterpossible scenarios, in some cases the relevant vicinity will be much larger than in others, but the process is the same.[9] One may be concerned that such a process will in fact have no end, and that as we are dealing with metaphysical necessity, there will always be contradictions in the counterpossible scenario we imagine. Alternatively, the concern may be that the process takes so long that in rejecting background assumptions we end up with a completely different arithmetic system in which everything works so differently that we cannot retrieve any useful conclusions from consideration of the scenario. Baron et al. (2017, p. 8) address such concerns by pointing out that a similar process occurs in the consideration of ordinary counterfactuals.

In ordinary counterfactuals, we may run into contradictions in considering the scenario, but we simply reject all and only those relevant for whatever our purposes may be. For example, in considering the case of whether Suzy's throwing of the rock caused the window to break, we may consider counterfactuals beginning "If Suzy had not thrown the rock…". In such cases there are of course inconsistencies, in the scenario we are considering it may be the case that Suzy indeed moved to throw the rock but that the rock did not move for some unspecified reason. Or it could even be that Suzy made the decision to move her arm but that it simply did not happen (Baron et al., 2017, p. 8). It simply is not the case that we go back through the entirety of history to make this scenario consistent. In fact we tend to ignore the inconsistencies and just conceptualise Suzy failing to throw the rock, without necessarily filling in the background details as to how

[8] Berto et al. (2018, p. 704). discuss a similar point related to Euclid's theorem. In the context of a reductio proof we should hold everything fixed, but in other contexts it might make sense to jettison some assumptions and in such cases not all statements would mathematically follow (i.e., some counterpossibles would be false).

[9] One concern I have with the specific way Baron et al. (2017) go about the process in their paper is that it seems they might in fact no longer be considering counterpossibles because they redefine what various mathematical operators mean, specifically addition. It seems at that stage that rather than considering impossible ways for the specific mathematical system we have to be, they might simply be considering a different mathematical system, and so this simply seems like a counterfactual. Compare this to a counterfactual like "Had the queen in chess not been able to move diagonally, then…" this does not seem to be a claim about the specific set of rules we have for chess currently, but rather about a different set of rules.

that failure was realised. It seems that the same process should take place in the countermathematicals case. Baron et al. (2017, p. 9) think that when we dispense with the immediate contradictions in the mathematical case we can leave it there and ignore the rest, even if actually addressing all the contradictions would be an infinite process. Now of course it might be the case that addressing the immediate contradictions in an ordinary counterfactual case is much simpler and a much smaller job than addressing the immediate contradictions in a mathematical counterpossible. But there is no reason to think that this process is anything more than a difference in degree. If we perform this process then we can consider internally consistent (but impossible) scenarios and try to determine what would/would not be the case, were these scenarios to take place.

3.5. Countermathematicals in Explanation

So far we have discussed why it is that we should judge scientific counterpossibles to be non-trivial. We have also shown how there are different uses of counterpossibles depending on which sort of reasoning we are engaged in (either discovering the truth value or reasoning on the supposition of truth regardless of the actual truth value). It is time to extend this to the mathematical case. We have already seen how counterpossibles play a role in the first kind of reasoning. When we aim to test a mathematical hypothesis we hold everything else fixed and see if we run into contradictions. If we do, then the antecedent is false. In such cases, it might turn out that all the countermathematicals involved are true. But importantly, they are not true because vacuism is correct, they are true because they follow from mathematical reasoning. Euclid's theorem discussed earlier was one example of this. It also seems plausible that Williamson's proof (2007; 2018), is an example of this kind of counterfactual reasoning. But countermathematicals can also be used in the second kind of reasoning process, to explain something in the world.

There are many examples of this, but a key one is the discussion by Lange (2017) about distinctively mathematical explanations.[10] Although this work of Lange's does not enter into these areas of counterpossible debate, I think it does bear upon it in a number of ways. One (very simple) example of a distinctively mathematical explanation would be something akin to "The reason that Jane cannot divide her 23 strawberries equally between her 3 children (without cutting), is because 23 is indivisible by 3". In context of the scientific explanations we considered earlier, this is quite similar to the explanation of why diamond does not conduct electricity. So, to put the mathematical explanation in counterfactual terms (as is legitimate practice) we can say "Had 23 been evenly divisible

[10] This kind of explanation is parallel to the usage of counterpossibles to explain the poor conductivity of diamond and the movement of water as described in Section 2.1. We might know that the antecedent is false, but we want to suppose it to be true to highlight some sort of dependence relation.

by 3, then Jane would have been able to divide her 23 strawberries evenly between her 3 children (without cutting)”. In the case of a counterfactual like this, we are not trying to discover the truth value of the antecedent. We know it is false, indeed we know it is impossible. What we are trying to do is work out what would happen if it were true. We have to suppose the antecedent to be true. In order to suppose it to be true, we simply cannot hold everything else fixed. When we start to jettison assumptions (for starters, we might get rid of the fact that 23 is prime), we will no longer run into a straightforward contradiction between the antecedent and consequent. Yli-Vakkuri and Hawthorne remark when discussing provability in mathematics that “[…] ‘⊢’ expresses provability in mathematics—by which we mean pure mathematics. $\Gamma \vdash A$ only if both A and all of the statements in Γ are pure mathematical statements” (2020, p. 560). When we are discussing counterpossibles which contain a non-mathematical consequent, the consequent will not follow mathematically from the antecedent. As such, the counterpossible as a whole may well turn out to be false. The mistake of the vacuist is in thinking that the first kind of reasoning process is the only one, or that it is the most important one. If it is the case that all the countermathematicals used in the first kind of process are true, it is not because of vacuism, it is because of mathematical practice and its results. In the second case, it is simply not the case that they all turn out true, their truth value will vary from world to world, just as with counterfactuals.

4. Potential Problems

4.1. Do Mathematicians Use Counterfactuals?

One general point to bring up is whether or not mathematics does indeed use counterfactuals, as opposed to merely appearing to use them through language choice but actually relying on something else.[11] Non-vacuists about countermathematicals clearly think that mathematics makes use of them. But it is important to point out that many prominent vacuists also think this. For example, as Yli-Vakkuri and Hawthorne say, “we will argue, mathematics makes use of the counterfactual conditional…” and that this usage “is by no means a marginal feature of mathematical discourse” (2020, p. 552). Indeed they themselves ultimately view it as indispensable. Perhaps the most vocal vacuist, Timothy Williamson, also concedes that we must account for the use of counterfactuals in mathematics as it is a legitimate practice (2018, p. 363). Reutlinger et al. (2020) began a more formal study of mathematical language and, from those preliminary results,[12] it seems to be the case that mathematicians frequently use counterfactuals. Now of course, one could maintain a commitment to this choice of language being a facade, perhaps disguising material conditionals. However,

[11] Thank you to a reviewer for bringing up the importance of clarifying this point.
[12] Available in Section 5 of that paper.

given the prevalence of seeming-counterfactuals in mathematics, and given that mathematicians seem to be taking themselves to be talking in counterfactual terms, this would be quite a revisionary view of mathematical practice. As such, I think it would require extensive independent justification to be considered as a serious objection. Whilst both vacuists and non-vacuists seem to be taking counterfactual usage for granted, I think we can simply assume the usage is genuine for the purposes of this debate.

4.2. How Do Mathematicians Use Countermathematicals?

Even granted that mathematicians genuinely appeal to countermathematicals in their writings, it is unclear how they are appealing to them, i.e., if they are appealing to them as vacuous or not. Williamson would clearly disagree with my claims that the judgements of mathematicians about specific countermathematicals would match the non-vacuist judgement. It is worth discussing some evidence in favour of the non-vacuist view. Yli-Vakkuri and Hawthorne (2020, p. 567) say that, in conversations with mathematicians, they will tend to assert counterpossibles like the following:

(TB): "If AC were false, then the Tarski-Banach theorem would not be provable from the truths of set theory"

whilst denying counterpossibles like:

(TB)[1]: "If AC were false, then the Tarski-Banach theorem would be provable from the truths of set theory".

I think this is exactly as the non-vacuist should accept (and indeed as I assert), and confusing only for the vacuist. The reason for this is that (TB) is true because the consequent would follow if the antecedent were true. Part of what is for AC to be false is for the Tarski-Banach theorem to fail to be provable from the truths of set theory.[13] Thus, (TB)[1] is false because if the axiom of choice were false, it would not be possible to prove the Tarski-Banach theorem from the truths of set theory, such a proof requires the truth of the axiom of choice. This element of mathematical practice is an anomaly for the vacuist, as noted by Yli-

[13] A good summary of this idea is available in Sendłak's (2021). Sendłak argues that counterpossibles such as "Had paraconsistent logic been true at the actual world then…" are paraphrases of statements like "According to the story of paraconsistent logic…". If the consequent in the paraphrase makes the statement false overall, then the counterpossible equivalent should also be false. By Sendłak's argument, Williamson (2020, pp. 129–130) is simply wrong when he accepts "...if the Bible is to be believed, there are angels" and also accepts "if the Bible is to be believed, there are no angels". Believing the antecedent to be false does not justify accepting the second statement as true, because that is simply false according to the story of the Bible.

Vakkuri and Hawthorne (2020, pp. 567–568). This practice also extends to logicians discussing counterlogicals (counterfactuals with a logically impossible antecedent). This practice which seems to contradict vacuism is a problem for vacuists to solve. If this practice is stable then vacuists will have to be quite radically revisionary about mathematical/logical practice, an obvious weakness. Non-vacuists, however, have a prima facie explanation of this phenomenon; the reason that mathematicians deny such counterpossibles is because such counterpossibles are false. In these cases, the consequent does not follow from the relevant antecedent.

Further support for the non-triviality of countermathematicals can be found in (Jenny, 2018). Jenny proposes that mathematical practice implicitly relies on the assumption that countermathematicals are non-trivial, specifically in the case of relative computability theory. This is important work. Jenny also proposes (2018, p. 552) a project going forward whereby non-vacuists should aim to find counterpossibles in other areas, such as the sciences, to defeat vacuism on multiple fronts. As Jenny says

> Once we have a clearer picture of the areas where non-vacuous counterpossibles are indispensable and once we have model theories for these various classes of counterpossibles, we may then investigate to what extent we can integrate these model theories to come up with a unified and fully general theory of non-vacuous counterpossibles. (Jenny, 2018, pp. 552–553)

My paper can then be seen as a continuation of the Jenny project, an attempt to bring counterpossibles in these distinct areas together. This is also where my paper goes further than Jenny. This paper aims not merely to show individual cases of non-trivial counterpossibles in distinct areas, but also to show why these are non-trivial. I aim to show the process we need to engage in to get the result of non-triviality, along with the fully general theory that Jenny is looking for. I think the only way for non-vacuists to make a start on this general theory is to highlight the mistakes that vacuists make by making clear the requirements to genuinely conceive of something, and show how vacuists fail to do this.

4.3. Is Williamson in Fact Genuinely Conceiving of a Distinct World?

One may object to my criticism that Williamson (2018) has not considered a distinct world, and has simply considered the actual world. One way to do this can be drawn out from the work of Yli-Vakkuri and Hawthorne (2020). Yli-Vakkuri and Hawthorne say take a standard proof by reductio in maths, e.g., Euclid's theorem. In this proof, one initially supposes that there is indeed a largest prime. Then, given this claim and other established truths, they deduce various other statements and eventually show that the hypothesis in question was false. The allegation would be that I have unfairly characterised the mathematical process, because the above describes a situation in which one does suppose the false hypothesis to be true. This is not quite right though, and in fact we can use more discussion from the Yli-Vakkuri and Hawthorne paper to explain this. In their paper, they make the distinction between a consensus and a non-consensus context. As they say:

> In a consensus context the relevant axioms are taken for granted, it is common ground that they are being taken for granted, and no one is interested in challenging any of the axioms or in exploring the ramifications of giving up some but not all of the axioms […]. In a non-consensus context one is not entitled to assume that all of the axioms are true and hence also not entitled to assume that they are provable, since provability entails truth. (Yli-Vakkuri, Hawthorne, 2020, p. 566)

What I think it takes to genuinely suppose a statement/hypothesis, is to be in a non-consensus context. For it is only in a non-consensus context that you drop the assumptions you have that will immediately contradict the hypothesis. In a consensus context, the countermathematicals may all turn out to be true, but once again not vacuously so, because they followed from the relevant mathematics. In a non-consensus context, this is not the case. When one jettisons assumptions, one will not immediately run into contradictions, so the truth value of the counterpossibles will be up for grabs. To decide whether or not Euclid's theorem is a case of a consensus/non-consensus context, let us reiterate what goes on in that example. As Yli-Vakkuri and Hawthorne (2020, p. 558) say, we take a set of assumed axioms, $\mathbf{\Gamma}$, e.g., the Peano axioms, and *A*, which is the claim that there is a largest prime, and ultimately conclude *B*, our desired contradiction which shows us that the claim, *A*, was false. We should be able to see that, in their own terms, this sounds like a consensus context because the set of assumptions, $\mathbf{\Gamma}$, have not been modified. This matters because $\mathbf{\Gamma}$ will either directly contain the proposition ~*A*, or ~*A* will be a logical consequence of $\mathbf{\Gamma}$. In this way, consensus contexts fail to be a genuine conception/supposition of *A* being the case, because they implicitly assume that ~*A* is the case.

To make clear the implications for Williamson's argument, my allegation is that Williamson stays within a consensus context. This is insufficient for a genuine conception/supposition of *A*. In terms of the ways we might use counterfactuals discussed in the introduction, this is the first kind of reasoning process, not the second. It is a consensus rather than non-consensus context; and as I have claimed, the second kind of process is the one which can produce false counterpossibles. There is further support for this later in the paper when Yli-Vakkuri and Hawthorne describe a fictional community of mathematicians, "[f]or example, if *A* is the claim that there is a largest prime number, the point, if any, of a Boxer's assertion of $A \mathrel{\Box\!\!\rightarrow} B$ will be to contribute to an explanation of why there is no largest prime number" (2020, p. 566). In order to show that *A* is not the case, they have to keep in place the assumptions that will contradict it. Plainly this will be a consensus context which fails to be genuinely conceiving of a situation in which *A* is the case.

4.4. Is Counterpossible Usage a Fringe Phenomenon?

One of Williamson's key arguments in favour of vacuism is that counterpossibles are a fringe phenomenon. This seems to be implicit in his discussion in a number of places:

> [In a discussion of counterlogicals] it would be naive to take appearances uncritically at face value in a special case so marginal to normal use of language, for example by offering them as clear counterexamples to a proposed semantics of conditionals […] it is good methodological practice to concentrate on conditionals with less bizarre antecedents in determining our best semantic theory of conditionals […]. (2020, p. 60)

> After all, once the impossibility of a supposition is recognized, continuing to work out its implications is typically a waste of time and energy. (2020, p. 234)

> In linguistic practice, counterpossibles are a comparatively minor phenomenon, which is one reason why it is implausible to complicate the semantics of modalized conditionals in natural language just to achieve a desired outcome for them […]. (2020, p. 262)

However, I would simply deny that these are in fact fringe cases of counterfactuals. As we have seen, vast portions of scientific reasoning contain counterpossibles; mathematicians and logicians seem to use countermathematicals/counterlogicals respectively; and to engage in meaningful debate in metaphysics, it seems we might need to use countermetaphysicals. Given the wide usage of counterpossibles in all these domains, it makes little sense to describe these as fringe cases. Counterpossibles are a significant datum, and a semantic theory needs to account for their usage in a way that is not revisionary to the vast areas of practice which employ them. If, as Williamson says, such counterpossibles present a problem for a standard semantic theory, then I think that is simply a reason to reject that particular semantic theory, rather than be revisionary to all this practice.

5. Conclusion

One straightforward and orthodox reading of counterpossibles implies that they are all trivially true. However, this conflicts with a lot of intuitions we might hold. Of course intuitions only take us so far because not everyone holds them. But there is also strong precedent in the sciences to treat counterpossibles non-trivially. One reason to do this is that it seems that in cases of non-trivially true counterpossibles, we can reason from the antecedent to the consequent in some way. In non-trivially false counterpossibles, the consequent does not follow in this way. When we reject the assumption that the antecedent is false, we can use counterpossible form to discover the counterfactual dependence at play. For example, that the microphysical structure of diamond is responsible for its poor electrical conductivity, or to reason about what would have been the case if something impossible was the case, e.g., if Bohr's theory of the atom had been correct, we would have observed electrons in such-and-such a way. This reasoning can go wrong when we make a mis-ascription as to what would have been the case, resulting in non-trivially false counterpossibles. Despite apparent surface level difficulties, we can also extend the same reasoning process to intuitively non-trivially true countermathematicals. This also gives us space to have non-trivially false countermathematicals, when this reasoning process goes wrong. To engage in this kind of reasoning in either case we may need to, on some level, genuinely conceive of an impossible world. To consider a counterpossible, $A_i > B$, we have to genuinely conceive of a world in which A_i is the case, in doing so we have to reject our assumptions to the contrary. When we do this, some counterpossibles will turn out true, and some will turn out false. In other words, vacuism about counterpossibles is false.

REFERENCES

Baron, S., Colyvan, M., Ripley, D. (2017). How Mathematics Can Make a Difference. *Philosophers' Imprint*, *17*(3), 1–19.

Berto, F., French, R., Priest, G., Ripley, D. (2018). Williamson on Counterpossibles. *Journal of Philosophical Logic*, *47*(4), 693–713.

Brogaard, B., Salerno, J. (2013). Remarks on Counterpossibles. *Synthese*, *190*(4), 639–660.

Emery, N., Hill, C. H. (2017). Impossible Worlds and Metaphysical Explanation: Comments on Kment's Modality and Explanatory Reasoning. *Analysis*, *77*(1), 134–148.

Grice, H. P. (1975). Logic and Conversation. In P. Cole, J. Morgan (Eds.), *Syntax and Semantics, Volume 3: Speech Acts* (pp. 41–58). New York: Academic Press.

Jenny, M. (2018). Counterpossibles in Science: The Case of Relative Computability. *Noûs, 52*(3), 530–560.

Lange, M. (2017). *Because Without Cause*. Oxford: OUP.

Leng, M. (2010). *Mathematics and Reality*. Oxford: OUP.

Lewis, D. (1973). *Counterfactuals*. Hoboken: Blackwell.

McLoone, B. (2020). Calculus and Counterpossibles in Science. *Synthese*, *198*(12), 12153–12174.

Putnam H. (1969) Is Logic Empirical? In R. S. Cohen, M. W. Wartofsky (Eds.), Boston Studies in the Philosophy of Science (vol. 5, pp. 216–241). Springer: Dordrecht. doi:10.1007/978-94-010-3381-7_5

Reutlinger, A., Colyvan, M., Krzyżanowska, K. (2020). The Prospects for a Monist Theory of Non-Causal Explanation in Science and Mathematics. *Erkenntnis*. doi:10.1007/s10670-020-00273-w

Sendłak, M. (2021). Counterpossibles, Story Prefix and Trivialism. *Synthese*, *199*, 7283–7301. doi:10.1007/s11229-021-03114-7

Stalnaker, R. C. (1968). A Theory of Conditionals. In W. Harper, R. Stalnaker, G. Pearce (Eds.), *Ifs* (pp. 41–55). Dordrecht: D. Reidel Publishing Company.

Tan, P. (2019). Counterpossible Non-Vacuity in Scientific Practice. *Journal of Philosophy*, *116*(1), 32–60.

Williamson, T. (2007). *The Philosophy of Philosophy*. Oxford: Blackwell.

Williamson, T. (2018). Counterpossibles. *Topoi, 37*(3), 357–368.

Williamson, T. (2020). *Suppose and Tell: The Semantics and Heuristics of Conditionals*. Oxford: OUP.

Wilson, A. (2021). Counterpossible Reasoning in Physics. *Philosophy of Science*, *88*(5). doi:10.1086/714706

Yagisawa, T. (2010). *Worlds and Individuals, Possible and Otherwise*. Oxford: OUP.

Yli-Vakkuri, J., Hawthorne, J. (2020). The Necessity of Mathematics. *Noûs*, *54*(3), 549–577.

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Article

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THE NATURE OF PROPOSITIONAL DEDUCTION— A PIAGETIAN PERSPECTIVE

SUMMARY: Logic was once thought to describe the laws of thought; however, a plurality of logics has now replaced classical logic, obscuring rather than clarifying the nature of deduction with an embarrassment of riches. In cognitive science, on the other hand, logic is not thought to be a psychological theory of human reasoning. Research on human reasoning has traditionally focussed on deduction, although human reasoning is thought to be much richer, and two competing theories dominated discourse in cognitive science—the syntactic, formal-rule, and the semantic, mental-model theory. Jean Piaget also proposed a psychological theory of reasoning, but, in contrast to these classical theories, he advocated an operatory theory. Deduction is an integral part of Piaget's theory, and, in this paper, I briefly outline Piaget's operatory theory of propositional reasoning before explicating the nature of deduction embodied in it. I conclude that the nature of propositional deduction according to Piaget lies in the interpropositional grouping, a natural structure at the heart of propositional reasoning constituted by a closed system of operations of thought regulated by laws of transformation. I then argue that the nature of propositional deduction lies specifically in the lattice constituted by the inclusion/order relations between the propositions of the interpropositional grouping. Piaget did not conceive of the interpropositional grouping as a logic; nevertheless, I wind up arguing that a logic conceived as Piaget intimated would complement the plurality of logics with a natural logic.

KEYWORDS: operations of thought, grouping, structure, propositional reasoning, propositional deduction, Boolean algebra, lattice, logic.

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1. Introduction

From a logical perspective, an inference can be analysed into input, premisses, and output, conclusions, and a rule of inference governing the transition from premisses to conclusions. Inferences are then valid if the transition from premisses to conclusions occurs according to the rules of inference, and deductive inferences, considered paradigmatic of rational thought, are those whose conclusions are necessarily true if the premisses are true. Whilst the characterisation of inferences may not be controversial, what logic is beyond the study of inferences is (Hintikka, Sandu, 2007, Section 1). It is widely accepted today that logic, once synonymous with classical logic, has branched into a plurality of approaches, characterisations, and often rival concepts of validity (Restall, Beall, 2000; 2001; Russell, 2019). When asked what logic is, modern logicians, in contrast to their pre-20th-century colleagues, thus flush with an "embarrassment of riches" (Hintikka, Sandu, 2007, p. 13; Jacquette, 2007, p. 3). Equally, philosophers, logicians, and psychologists who seek the nature of deduction in logic alone (George, 1997; Posy, 1997) are chagrined.

Highlighting a mismatch between classical logic and human reasoning, Johnson-Laird concludes that "[l]ogic is an essential tool for all sciences, but it is not a psychological theory of reasoning" (Johnson-Laird, 2006, p. 17). Moreover, human reasoning is not thought to be synonymous with deduction (Harman, 1984; 1986; van Benthem, 2007; 2008); nevertheless, psychological research has tended to focus on deductive inferences (e.g., Johnson-Laird, 2006, p. 3). Broadly, cognitive scientists entertained three psychological theories of deductive reasoning: deduction is either a process based on factual knowledge; a syntactic, formal process, or a semantic process based on mental models (Johnson-Laird, 1999). Since the knowledge-based theory of deductive reasoning relies on memory of prior inferences, it is unable to account for inferences that are confidently made without precedents or even prior knowledge of the subject matter involved. Apart from tailor-made theories for particular aspects of reasoning, discourse on human reasoning was therefore portrayed as a two-horse race between the syntactic, formal-rules and semantic, mental-model theories (e.g., Byrne, Johnson-Laird, 2009; Johnson-Laird, 1999; 2006, Chapter Introduction; Rips, 2008).

In essence, advocates of formal-rule theories maintained that reasoners recognise logical forms in premisses and apply rules of inference akin to logical rules when inferring. Clearly, logic is the source of inspiration for these theories, and, rather than being an embarrassment, the plurality of logics could serve as a rich source of hypotheses for the formal rules of inference employed in reasoning (e.g., Stenning, van Lambalgen, 2008; 2011). Advocates of the mental-model theory, on the other hand, maintained that reasoning is a process of envisaging possibilities. In essence, reasoners construct models of possibilities consistent with the premisses given and draw conclusions based on them. In contrast to the syntactic, formal-rules theories and classical logic, content and context rather than form therefore play an important role in the mental-model theory. However, portraying the discourse as a two-horse race between rival psychological theories of reasoning was misleading. Jean Piaget also proposed a psychological theory of human reasoning, and, being based on operations of thought, it is fundamentally different from both the formal-rules and mental-model theories.

Piaget's theory has all but disappeared from mainstream debate on reasoning. A reason for its disappearance may lie in Piaget's theory being classified as an outdated progenitor of formal-rule theories (e.g., Johnson-Laird, 1999, p. 114; 2006, p. 14); "reasoning is nothing more than the propositional calculus itself" (Inhelder, Piaget, 1958, p. 305; Johnson-Laird, Byrne, Schaeken, 1992, p. 418), for example, is the citation Johnson-Laird uses to support a formal-rule-theory interpretation of Piaget's theory. However, Johnson-Laird does concede that "Piaget's views on logic are idiosyncratic" (Johnson-Laird, Byrne, Schaeken, 1992, p. 418), and "[i]t is not always easy to understand Piaget's theory" (Johnson-Laird, 2006, p. 249). Johnson-Laird's confessions express popular assessments of Piaget's theory of reasoning among Anglophone cognitive psychologists (Bond, 1978; 2005), and they corroborate Piaget's own impression that his work was poorly understood (Smith, Mueller, Carpendale, 2009, pp. 1–10).

Difficulties in understanding Piaget's theory are exacerbated by the inaccessibility of his original works in a predominately Anglophone research environment. He wrote in French, and translations into English are selective and not rarely dubious in quality (Smith, Mueller, Carpendale, 2009, pp. 28–44). Returning to the citation Johnson-Laird uses to substantiate his claim, Lesley Smith considers "reasoning is nothing more than the calculus embodied in propositional operations" (Smith, 1987, p. 344) to be a more faithful rendition of the French original. The difference in translations may seem minimal, but this paper shows that the operatory standpoint is essential for the correct understanding of Piaget's theory of human propositional reasoning and the nature of propositional deduction in particular.[1]

I begin my exposition of the operatory nature of deduction by introducing operations of thought (Section 2). Piaget attempted to cast the calculus embodied in propositional operations in a formal language by using the algebraic tools logic put at his disposal, and, due to its formal appearance, the calculus might easily be mistaken for a logic. Before setting out the interpropositional grouping, I therefore briefly explicate "psycho-logic" (Section 3) to clarify Piaget's intentions. I then go on to set out the calculus embodied in propositional operations of thought (Section 4), beginning with the most elementary interpropositional grouping constituted by the affirmation and negation of a single proposition (4.1) before setting out its systematic extensions to multiple propositions. At this point, I would like to apologise to the reader for rehearsing Piaget's operatory theory of reasoning in such detail, especially to those familiar with his work.

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[1] For the sake of brevity, the additional attributes are assumed on writing "reasoning" and "deduction" from now on.

In view of misconceptions surrounding Piaget's theory, however, I feel obliged to adumbrate and justify my interpretation.[2] The extensions are based on implication, and I set out the four distinct forms of implication inherent in the interpropositional grouping (Sections 4.2 and 4.3), before presenting Piaget's analysis of these forms from the point of view of deduction (Section 5). I then characterise the nature of deduction according to Piaget—broadly first, from a diachronic then a synchronic perspective; in detail second, by focusing specifically on the forms of implication (Section 6)—before concluding (Section 7) with a brief remark on a ramification of the nature of deduction according to Piaget for logical pluralism.

2. Operations

By joining propositions to others using propositional connectives, such as "and" (∧), "or" (∨), "if-then" (⊃), etc., compound propositions are constructed. The meaning of the compound proposition is then constituted by the meanings of the constituent propositions and the propositional connective involved. Just as compound propositions are composed of parts, the propositions themselves are also composite in nature. In contrast to compound propositions, however, the constituent parts are not propositions; "Mammals are vertebrates", for example, has a subject "mammals", predicate "vertebrates" and a logical constant "is". These parts can be substituted for others, and the meaning of the whole proposition is constituted by the meanings of its parts. Operations are intellectual activities that compose and decompose such connections between propositions or between the parts of propositions (Piaget, Grize, 1972, p. 9). Piaget denotes the former "interpropositional operations" and the latter "intrapropositional operations" (Piaget, Grize, 1972, pp. 34–35). In this paper, I focus on interpropositional operations although deduction also occurs in intrapropositional reasoning.

Whether intra- or interpropositional, Piaget describes the psychological nature of intellectual operations as follows:

> [O]perations are actions which are internalizable, reversible, and coordinated into systems characterized by laws which apply to the system as a whole. They are actions, since they are carried out on objects before being performed on symbols. They are internalizable, since they can also be carried out in thought without los-

[2] Partly due to no standard edition of Piaget's work in French existing and reliable English translations being few and far between, misconceptions of Piaget's work are abound. With readers' convenience in mind, some exegetical overlap is therefore inevitable (e.g., Winstanley, 2021). However, the manuscripts pursue entirely different questions on the basis of the exegeses: using Piaget's theory of reasoning as an example of how psychology may legitimately serve as logical evidence for logical theory, Winstanley (2021) focuses on the interface between logic and psychology and elaborates ramifications for anti-exceptionalism about logic; the current paper, in contrast, elucidates the nature of propositional deduction according to Piaget and therefore has a psychological focus.

> ing their original character of actions. They are reversible as against simple actions which are irreversible. In this way, the operation of combining can be inverted immediately into the operation of dissociating, whereas the act of writing from left to right cannot be inverted to one of writing from right to left without a new habit being acquired differing from the first. Finally, since operations do not exist in isolation they are connected in the form of structured wholes. (Piaget, 1957, p. 8; see also Piaget, 1971a, pp. 21–22; 2001, Chapter 2; Piaget, Beth, 1966, p. 172; Piaget, Grize, 1972, p. 55)

Piaget used logical tools to represent the structured wholes formed by operations. However, precisely because of their formal appearance, it is important to nip misconceptions in the bud by clarifying how Piaget intended these models to be understood.

3. Psycho-Logic

Logic is concerned with what conclusions follow from what premisses, and it develops techniques for determining the validity of inferences. Piaget's theory, on the other hand, is not primarily concerned with logical consequence, and it does not provide techniques for assessing the validity of arguments (Grize, 2013). Piaget understood his theory in analogy to mathematical physics. Physics investigates the physical world experimentally, and its criterium for truth is correspondence with empirical facts; mathematics, on the other hand, is neither based on experiment nor does its truth depend on agreement with empirical facts. It is a formal science whose truth depends solely on the formal consistency of the deductive systems constructed. Drawing on both deductive and empirical sources, mathematical physicists, aiming to understand the physical world, apply mathematics to physics to construct deductive theories based on the experimental findings of physics. Like mathematical physics, Piaget (1957; see also Bond 1978; 2005) also envisages "psycho-logic" or "logico-psychology" as a *tertium quid*. On the one hand, psychology investigates mental life empirically, and its criterion for truth is correspondence with experimental facts; on the other hand, logic, like mathematics, is a deductive science concerned with formal rigour rather than correspondence with facts, and it has developed algebraic techniques. Psycho-logic is an application of the algebraic tools of logic to the findings of experimental psychology, and it aims to construct a formal theory based on the experimental facts of psychology. In other words, psycho-logic uses logic to model the structured wholes systems of operations form.

In the next section, the most elementary model of interpropositional operations is set out first, and it is followed by progressive extensions.

4. The Interpropositional Grouping

Piaget modelled the structured wholes operations of thought form with a grouping. Roughly, a grouping is a structure incorporating the reversible operations of its namesake, mathematical groups, and the non-reversible operations of lattices.[3]

4.1. A Proposition and its Grouping

Given a single proposition p and its negation $\bar{p}$, Piaget and Grize (1972, pp. 321–322) set out the operations of the grouping as follows:

(i) The direct operation unites p disjunctively ($\vee p$) with other propositions of the system. Since $\bar{p}$ is currently the only other proposition, $\bar{p} \vee p = T$, for example; T is, therefore, a compound proposition comprised of p and $\bar{p}$, and it is also part of the system.

(ii) The inverse operation unites the negation of p conjunctively ($\wedge \bar{p}$) with other propositions of the system; for example, $T \wedge \bar{p} = \bar{p}$, $p \wedge \bar{p} = o$, etc. o is therefore also a proposition of the system.

(iii) The general identity operation, denoted ($\vee o$), is a) the product of direct and inverse operations, $p \wedge \bar{p} = o$; and b) it leaves the propositions it is composed with unaltered; for example, $p \vee o = p$, $\bar{p} \vee o = \bar{p}$, $T \vee o = T$, $o \vee o = o$, etc.

(iv) Despite not being composed of direct and inverse operations, some operations also leave the propositions they are composed with unaltered much like the general identity; for example, $p \vee p = p$; $\bar{p} \vee \bar{p} = \bar{p}$; $\bar{p} \wedge \bar{p} = \bar{p}$; etc.

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[3] Mathematically, a group is a set of elements with a binary operation that combines any two elements of the set to form a third in such a way that the group axioms— associativity, identity, and invertibility—are satisfied.

A lattice, on the other hand, can be defined in two different ways. On the one hand, it is a partially ordered set in which any two elements have both a least upper bound (meet) and a greatest lower bound (join). A partially ordered set, poset $\mathcal{P}$ for short, is an algebraic system in which a binary relation $x \geqq y$ is defined satisfying the following postulates:

P_1:	For all x, $x \geqq x$	(reflexive property)
P_2:	$x \geqq y \wedge y \geqq x, x = y$	(antisymmetric property)
P_3:	$x \geqq y \wedge y \geqq z, x \geqq z$	(transitive property)

The binary relation satisfying these postulates is called an inclusion or order relation (Rutherford, 1966, p. 1). For elements x and y of a lattice $\mathcal{L}$, the meet is denoted $x \cup y$ and the join $x \cap y$.

Alternatively, a lattice is a set $\mathcal{L}$ of elements with two binary operations $\cap$ and $\cup$ satisfying the following postulates for all $x, y, z, \ldots$ of $\mathcal{L}$ (Rutherford, 1966, pp. 4–5):

$L_{1\cap}$: $x \cap y = y \cap x$	$L_{1\cup}$: $x \cup y = y \cup x$	(Commutative Laws)
$L_{2\cap}$: $x \cap (x \cap z) = (x \cap y) \cap z$	$L_{2\cup}$: $x \cup (y \cup z) = (x \cup y) \cup z$	(Associative Laws)
$L_{3\cap}$: $x \cap (x \cup y) = x$	$L_{3\cup}$: $x \cup (x \cap y) = x$	(Absorptive Laws)

Via the identity $y = x \cap y \equiv x \geqq y \equiv x \cup y = x$, the two definitions can be shown to be equivalent (Rutherford, 1966, Section 4).

(self-inclusions), and $\bar{p} \vee T = T$, i.e., $p \vee (p \vee \bar{p}) = (p \vee \bar{p})$ (absorptions). These operations are *special identities*.

(v) Finally, the compositions are only partially associative; e.g., $p \vee (\bar{p} \vee o) = (p \vee \bar{p}) \vee o = T$, whereas $p \vee (p \wedge \bar{p}) \neq (p \vee p) \wedge \bar{p}$ because $p \vee (p \wedge \bar{p}) = p \vee o = p$ and $(p \vee p) \wedge \bar{p} = p \wedge \bar{p} = o$.[4]

The first three operations are reversible like the operations of a group. Moreover, $T = p \vee \bar{p}$, $p \wedge \bar{p} = o$, as well as $p \vee o = p$ and $\bar{p} \vee o = p$; the group-like operations are therefore reminiscent of the laws of thought, excluded middle, non-contradiction, and the law of identity, respectively. The fourth operation, on the other hand, is characteristic of a lattice, and $p \vee p$ (self-inclusion) and $p \vee o$ (direct operation and the general identity), especially, limit the associativity characteristic of the operations of a group.

Since the direct operation operates on all the propositions of the system, it also combines $\bar{p}$ disjunctively with other propositions of the system so that $p \vee \bar{p} = T$, for example, and the corresponding inverse operation is $T \wedge \bar{\bar{p}} = T \wedge p = p$. Since the inverse operation $\overline{(\vee\bar{p})} = \wedge\bar{\bar{p}} = \wedge p$, conversely $\overline{(\wedge p)} = \overline{(\wedge \bar{\bar{p}})} = \overline{\overline{(\vee \bar{p})}} = (\vee \bar{p})$, Piaget and Grize (1972, pp. 321–322) define $\wedge p$ and $\vee\bar{p}$ as another reversible pair of operations in the grouping just like $\vee p$ and $\wedge \bar{p}$. These operations introduce their own special identities; for example, $p \wedge p = p$ (self-inclusion), but $p \wedge T = p$ and $\bar{p} \wedge T = \bar{p}$ instead of absorption. $\wedge T$ is the most general of these special identities, and, like the general identity operation, it leaves the propositions it is composed with unaltered; unlike the general identity, however, it is not composed of direct and inverse operations.

4.2. The Forms of Implication

Implication is one of the few logical operators already present in the elementary grouping involving the affirmation and negation of a single proposition (Piaget, Grize, 1972, p. 323),[5] and, by differentiating the implication $p \supset T$ into a chain of implications, the elementary grouping can be extended to multiple propositions as follows:

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[4] Another possible source of confusion needs to be nipped in the bud. The symbolism is familiar from propositional logic; however, it does not have the conventional meaning (Apostel, 1982). Piaget (Piaget, Beth, 1966, pp. 180–181) simply found it convenient to adopt the symbolism of propositional logic and give it a whole new meaning in the context of his operatory theory.

[5] $(T \wedge p) \vee (T \wedge \bar{p})$ is a composition of the operations of the elementary grouping. By substituting the p and T of the elementary grouping for p and q in column 7 of Table 1, the disjunctive normal form of the conditional $p \supset T = (T \wedge p) \vee (T \wedge \bar{p}) \vee (\bar{T} \wedge \bar{p})$ is obtained. However, it reduces to $p \supset T = (T \wedge p) \vee (T \wedge \bar{p})$ in the elementary grouping since $\bar{T} = o$ therefore $(\bar{T} \wedge \bar{p}) = o$.

$p \supset q; q \supset r; r \supset s; s \supset t \ldots u \supset T$

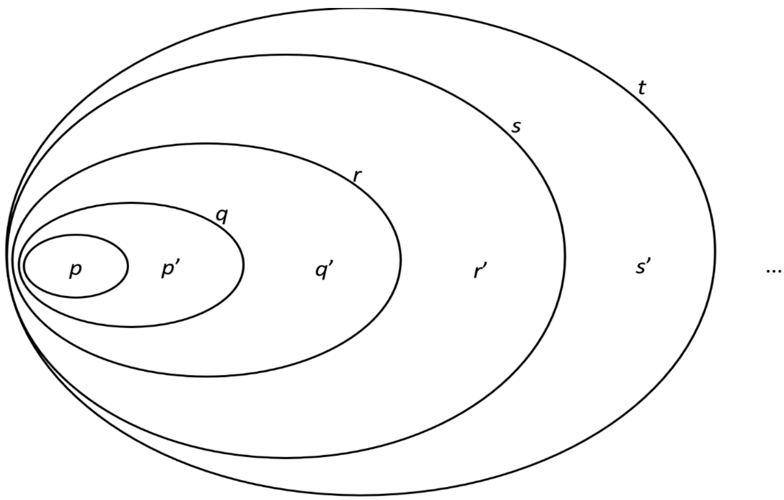
where q now plays the same role for p as T did for p in the elementary grouping; r, for q; s, for r, etc. By systematically elaborating the operations of the grouping on this chain of implications, Piaget and Grize (1972, pp. 324–325) discerned four distinct forms of implication.

4.2.1. Form I.

In Form I (Piaget, Grize, 1972, pp. 324–327), $q = pq \vee \bar{p}q$ expresses the common and non-common parts of p and q in analogy with $T = (T \wedge p) \vee (T \wedge \bar{p})$. The non-common part, $\bar{p}q$, is the relative complement of p in q, and Piaget denotes it p'; q can therefore be expressed more concisely as $q = p \vee p$'. Proceeding analogously for the other propositions in the chain, we have $r = q \vee q$', where $q' = r \wedge \bar{q}$; $s = r \vee r'$, where $r' = s \wedge \bar{r}$; etc. (see Figure 1).

Figure 1

Grouping of Implications—Form I



Note. Piaget calls $p, q, r, s, t, \ldots$ primary propositions and their relative complements $p' = q \wedge \bar{p}, q' = r \wedge \bar{q}, r' = s \wedge \bar{r}, \ldots$ secondary propositions. Primary propositions in the hierarchy are composed of the primary and secondary propositions of the previous level as follows: $q = p \vee p', r = q \vee q', s = r \vee r', \ldots$ (Piaget, Grize, 1972, p. 324, Fig. 46).

Using $\vee p$ and $\wedge \bar{p}$, one of the reversible pairs of operations of the elementary grouping (see Section 4.1), Piaget shows that Form I also constitutes a grouping:

(i) The direct operation $\vee p$ composes a proposition p with another proposition of the system to form an equivalence; e.g., $p \vee p' = q$; $q \vee q' = r$; etc.

(ii) The inverse operation $\wedge \bar{p}$ composes the negation of a proposition conjunctively with another proposition of the system; e.g., $q \wedge \bar{p} = p'$; $q \wedge \bar{p}' = p$; $\bar{p} \wedge \bar{p}' = \bar{q}$; $r \wedge \bar{p} = p' \vee q'$; etc.[6]

(iii) The general identity operation $\vee o$ is the product of the direct and inverse operations, e.g., $p \wedge \bar{p} = o$. Composed with other operations, the general identity leaves them unchanged; e.g., $p \vee o = p$; $\bar{p} \vee o = \bar{p}$, etc.

(iv) The special identities are self-inclusions; e.g., $p \vee p = p$, $\bar{p} \vee \bar{p} = \bar{p}$, $\bar{p} \wedge \bar{p} = \bar{p}$, etc.; and absorptions; e.g., $p \vee q = q$.[7]

(v) Associativity is limited because of the special identities.

As well as being a multipropositional differentiation of the implication present in the most elementary grouping involving the affirmation and negation of a single proposition, Form I thus also constitutes a grouping with $\vee p$ and $\wedge \bar{p}$ as direct and inverse operations (Piaget, Grize, 1972, pp. 324–325). Moreover, it is analogous to the inclusion of classes $P \subset Q \subset R \subset S \subset$, etc., familiar from biological taxonomies, genealogies, etc. Piaget (Piaget, Grize, 1972, p. 103) call *S*, *P*, *Q*, *R*, *S*, etc. *primary* classes, and these primary classes have relative complements *P*', *Q*', *R*' etc., which he calls *secondary* classes. The grandchildren of a grandparent *Q*, for example, are comprised of the children of one of *Q*'s children *P*, and their first cousins *P*'. In terms of primary and secondary classes, the classes constituting the nesting inclusions are therefore as follows:

$$P \cup P' = Q,\ Q \cup Q' = R,\ R \cup R' = S, \text{ etc.}$$

Let propositions *p*, *q*, *r*, *s*, etc. express the membership of an element *x* in the primary classes *P*, *Q*, *R*, *S*, etc. and *p*', *q*', *r*', etc., its membership in the secondary classes *P*', *Q*', *R*', etc. Clearly, if *q* is true, *x* is a member of $Q = P \cup P'$ therefore *x* is a member of either *P* or *P*', i.e., $p \vee p'$; Form I, therefore, corresponds to the nesting inclusions of classes typically found in Porphyrian trees. In fact, the intrapropositional operations on such classes also constitute groupings

[6] a) Unlike classical logic, negation of a single proposition is not a unary operator; it is equivalent to an inverse operation and therefore a binary operator; $\bar{p}$, for example, is equivalent to $\wedge \bar{p}$, i.e., $\bar{p} = T \wedge \bar{p}$, the relative complement with respect to *T*. $\bar{\bar{p}} = p$ is therefore equivalent to $\overline{(\wedge \bar{p})} = p$, the complement of the complement of *p* with respect to *T*, rather than $\bar{p} \wedge \bar{p} = \bar{p}$.

b) Moving a proposition from one side of an equivalence to the other is equivalent to applying the inverse operation; e.g., if $p \vee p' = q$, then $(p \vee p') \wedge \bar{p}' = q \wedge \bar{p}'$, i.e., $p = q \wedge \bar{p}'$; similarly, $p' = q \wedge \bar{p}$; $(p \vee p') \wedge \bar{q} = o$, etc.

[7] Due to the special identities, rules of composition must also be observed when propositions are transferred across equivalences; e.g., $(p \vee p) = p$ cannot become $p = (p \wedge \bar{p})$ when transferring $\vee p$ from the left to $\wedge \bar{p}$ on the right since $(p \wedge \bar{p}) = o$ and $p \neq o$.

(Piaget, Grize, 1972, Chapter II), and Form I models one of these groupings in terms of propositions (Piaget, Grize 1972, p. 324). For the present purposes, however, the correspondence with nesting inclusions of classes simply facilitates the recognition of valid inferences. Clearly, if x is a member of a class, it is automatically a member of all of its superclasses; primary p, q, r, etc. and secondary p', q', r', etc. propositions, therefore, imply primary propositions of higher rank; e.g., $p' \supset t$ or $r \supset u$, etc. Conversely, if x is a member of a primary class, it must be a member of one of the disjoint classes composing it; each primary proposition therefore implies those propositions composing it but as a whole; e.g., $s \supset (p \vee p' \vee r')$. Finally, any subclass of a primary class of higher order can be determined by eliminating relative complements; any proposition can therefore be inferred from those of higher rank by negating complementaries; e.g., $q' = t \wedge \overline{s}' \wedge \overline{r}' \wedge \overline{q}$ (Piaget, Grize, 1972, p. 326). As well as the membership of elements in classes, propositions can also represent relations.

4.2.2. Form II.

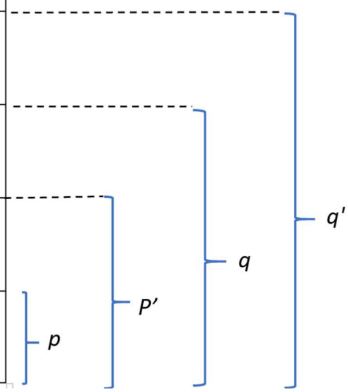
Like the pair $\vee p$ and $\wedge \overline{p}$, $\wedge p$ and $\vee \overline{p}$ are also reversible operations of the elementary grouping, and Piaget (Piaget, Grize, 1972, pp. 327–329) based the second form of implications on this pair. As in Form I, relations between the propositions o, p, $\overline{p}$, and T are generalised to multiple propositions; however, Form II focuses on the conjunctive rather than the disjunctive compositions uniting propositions into a whole. With $\wedge p$ as the direct operation, a series of compound propositions can be constructed by composing propositions p, q, r, etc. with other propositions of the system p', q', r', etc. conjunctively to obtain the following equivalences: $p \wedge p' = q$; $q \wedge q' = r$; $r \wedge r' = s$; etc. (see Figure 2 on the next page).

Unlike Form I, Form II does not correspond to intrapropositional operations on classes. Grandchildren, for example, cannot simultaneously be siblings and their own first cousins since the intersection of complementary classes is empty. Nevertheless, elements of classes equivalent from one point of view may differ in degrees of a common property, thus allowing them to be ordered. Siblings A, B, C, etc., for example, differ according to age, and it is possible to order them via the order of birth without knowing their exact numerical ages: If A was born before B, A is older than B ($A \rightarrow B$), and if B was born before C, B is older than C ($B \rightarrow C$); clearly, A was born before C so that A is older than C ($A \rightarrow C$). In terms of propositions, let p represent "$A \rightarrow B$" and p' represent "$B \rightarrow C$", then q would represent "$A \rightarrow C$". In contrast to Form I, in which it is possible to infer q alone from either of the parts p or p' constituting it, neither p nor p' are sufficient by themselves to infer q in Form II. Just as it is not possible to infer A is older than C ($A \rightarrow C$) on the basis of either A being older than B ($A \rightarrow B$) or B being older than C ($B \rightarrow C$) alone, only p in conjunction with p' allows q to be inferred.

Figure 2

Grouping of Implications—Form II

$t=$	$s \wedge s'$	s'
$s=$	$r \wedge r'$	r'
$r=$	$q \wedge q'$	q'
$q=$	$p \wedge p'$	p'
$p=$	p	p



Note. In the rows of the middle column, the compound propositions $p \wedge p$', $q \wedge q$', $r \wedge r$', etc. are formed by conjunctions of the propositions in the rows immediately below it and the proposition to its right; for example, q (= $p \wedge p$') is the conjunction of p, below, and p', to the right; r (= $q \wedge q$'), of q (= $p \wedge p$') below, and q' to the right; etc. (Piaget, Grize, 1972, Fig. 47).

Conversely, maintaining "$A \rightarrow C$" (q) while denying either "$B \rightarrow C$" (p') or "$A \rightarrow B$" (p) would be contradictory since it affirms the whole relation while denying one of its constituent parts. Analogously, $p = q \wedge \bar{p}$' and $p' = q \wedge \bar{p}$ would simultaneously assert the truth of q and the falsity of one of its constituent parts since $q = p \wedge p$'. The inverse operation used in Form I of the interpropositional grouping cannot, therefore, serve as an inverse operation in this form. The disjunctive composition of the negation of a proposition ($\vee \bar{p}$) with another proposition of the system, on the other hand, can, and compositions with this operation are $q \vee \bar{p}' = p$; $q \vee \bar{p} = p'$; $\bar{p} \vee \bar{p}' = \bar{q}$, etc., for example.

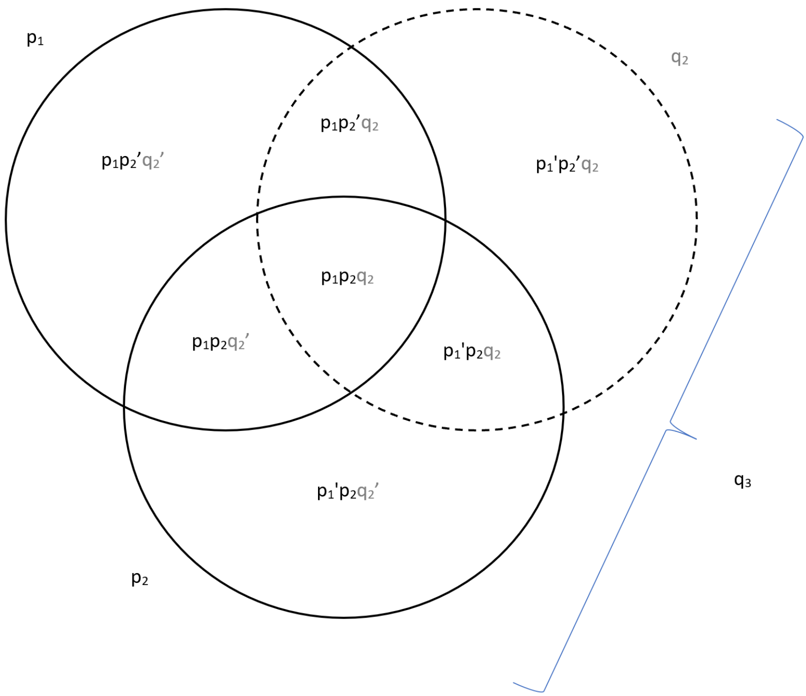
4.2.3. Form III.

In Form I, the wholes are constituted by exclusive disjunctions $p \vee p' = q$, $q \vee q' = r$, etc. The parts constituting the whole, therefore, have nothing in common. In Form II, on the other hand, the wholes are constituted by a conjunction $p \wedge p' = q$, $q \wedge q' = r$, etc.; the whole is therefore constituted by what its parts have in common. Whilst Forms I and II are both extensions of the elementary grouping, they also represent extremes since the wholes are comprised of either entirely disjoint, Form I, or entirely conjoint, Form II, propositions. By constituting the wholes with propositions that are neither entirely disjoint nor entirely conjoint, Form III lies between these extremes.

Let propositions p_1 and p_2 constitute the whole q_1 via a non-exclusive disjunction $p_1 \lor p_2 = q_1$. Like the previous forms, Form III also introduces new implications: $\bar{p}_1 \supset p_2$ and $\bar{p}_2 \supset p_1$ (see Figure 3).

Figure 3

Grouping of Implications—Form III



Note. q_1 is the whole constituted by the non-exclusive disjunction of two propositions p_1 and p_2, and it is comprised of the three disjoint parts p_1p_2', p_1p_2 and p_1'p_2, where p_1' = $\bar{p}_1 \land q_1$ and p_2' = $\bar{p}_2 \land q_1$. Similarly, r_1 is not included in the diagram, but it designates the whole constituted by the non-exclusive disjunction $q_1 \lor q_2$, and is therefore comprised of the 7 disjoint parts indicated. The shade of the font indicates the origin of the contributions of the parts. Although the hierarchy of nesting propositions continues, a two-dimensional representation of their partitions has reached its limit (for a schematic representation, see Piaget, Grize, 1972, Fig. 48). $q_3 = p_2 \lor q_2$ does not belong to the hierarchy directly; however, it highlights a part-whole relation inherent in the nesting hierarchy of propositions that is relevant to the axiomatisation of propositional logic.

By defining p_1' as the proposition ($\bar{p}_1 \land q_1$) and p_2' as ($\bar{p}_2 \land q_1$), i.e., as relative complements, the grouping is as follows:

(i) The direct operation constitutes the following nesting wholes:

$(p_1 \vee p_2) = q_1$
$(q_1 \vee q_2) = (p_1 \vee p_2 \vee q_2) = r_1$
$(r_1 \vee r_2) = (p_1 \vee p_2 \vee q_2 \vee r_2) = s_1$
$(s_1 \vee s_2) = (p_1 \vee p_2 \vee q_2 \vee r_2 \vee s_2) = t_1$, etc.

Each of these wholes is composed of three disjoint parts (see Figure 3):

$q_1 = (p_1 \wedge p_2) \vee (p_1 \wedge p_2') \vee (p_1' \wedge p_2)$
$r_1 = (q_1 \wedge q_2) \vee (q_1 \wedge q_2') \vee (q_1' \wedge q_2)$, etc.

(ii) And conjunctions of negations of these parts constitute inverse operations, e.g.:

$p_1 = q_1 \wedge \overline{(p_1' \wedge p_2)}$; $p_2 = q_1 \wedge \overline{(p_1 \wedge p_2')}$

(iii) The general identity is, for example:

$q_1 \wedge \bar{q}_1 = o$; i.e., $(p_1 \vee p_2) \wedge \overline{(p_1 \vee p_2)} = (p_1 \vee p_2) \wedge (\bar{p}_1 \wedge \bar{p}_2) = o$

(iv) The special identities are, for example:
$p_1 \vee p_1 = p_1$; $q_1 \vee q_1 = q_1$; $p_1 \vee q_1 = q_1$

Piaget illustrates Form III in analogy with classes. Let P_1 be a class of blood relatives and P_2 be a class of relatives by marriage. Forming the union of $P_1 \cup P_2 = Q_1$, an individual belonging to Q_1 can be a blood relative and an in-law or one without the other. p_1 = "$x \in P_1$", p_2 = "$x \in P_2$" and q_1 = "$x \in Q_1$" express memberships propositionally, and, one of the members of Q_1 marrying, a new class of in-laws Q_2 is formed, in which some are blood relatives and in-laws while others are one without the other. The corresponding proposition is q_2 = "$x \in Q_2$", and $(q_1 \vee q_2) = r_1$ represents the union of these classes $Q_1 \cup Q_2$. Continuing in this vein, classes corresponding to s_1, t_1, etc. can be constructed.

Piaget highlights some implications in Form III and draws particular attention to one by defining q_3 as $p_2 \vee q_2$ (see Figure 3). In terms of the corresponding classes, it is clear that $P_2 \subset Q_3$. Although $P_2 \cup P_1$ is no longer included in Q_3, it is nevertheless included in $Q_3 \cup P_1$, the enlargement of Q_3 by the same class P_1; hence $(P_2 \cup P_1) \subset Q_3 \cup P_1$ provided $P_2 \subset Q_3$. Translated into propositions, $p_2 \vee p_1 \supset q_3 \vee p_1$ provided $p_2 \supset q_3$, and, through suitable substitutions, this formula is recognizable as $(p \supset q) \supset [(r \vee p) \supset (r \vee q)]$, axiom IV of Bertrand Russell's axiomatisation of propositional logic. According to Piaget, the special identity of the grouping $(p \vee p) = p$ also comes to expression in $(p \vee p) \supset p$, axiom I; axiom II, $p \supset (p \vee q)$, expresses the inclusion of parts in the whole

$(p \vee p') = q$ as well as special identities due to absorption $(p \vee q) = q$; and axiom III, $(p \vee q) \supset (q \vee p)$, expresses the commutativity of the operation $\vee$, on which the Forms I and III of the grouping rest. The Forms I and III of the grouping of implications thus condense the axioms of propositional logic, according to Piaget (Piaget, Grize, 1972, p. 331).

4.2.4. Form IV.

Although the forms of implication already presented are sufficient for an axiomatization of propositional logic, there is nevertheless a fourth form (Piaget, Grize, 1972, pp. 331–334). In Form I, the wholes $q = p \vee p'$, etc. are comprised of two disjoint parts; in Form III, on the other hand, the wholes are comprised of three disjoint parts $q = (p_1 \wedge p_2) \vee (p_1 \wedge p_2') \vee (p_1' \wedge p_2)$. Form IV complicates matters still further by adding yet another disjoint part $(p_1' \wedge p_2')$ so that the whole is now constituted by four disjoint parts:

$$q = (p_1 \wedge p_2) \vee (p_1 \wedge p_2') \vee (p_1' \wedge p_2) \vee (p_1' \wedge p_2').$$

However, Form IV is not just a complication for complication's sake. Class Q corresponding to the whole q in Form I has many alternative partitions; for example, Europeans (Q) are, from a German perspective, either Germans (P_1) or non-Germans (P_1'); from an Austrian point of view, on the other hand, they are Austrians (P_2) and non-Austrians (P_2'). Consequently, some Germans are non-Austrians and some Austrians are non-German. Since dual nationality is possible in the European Union, there are therefore German Austrians ($P_1 \cap P_2$), Germans who are not also Austrians ($P_1 \cap P_2'$), Austrians who are not also Germans ($P_1' \cap P_2$), and Europeans who are neither Austrian nor German ($P_1' \cap P_2'$). Analogously, four disjoint parts constitute the whole in Form IV: $q = (p_1 \wedge p_2) \vee (p_1 \wedge p_2') \vee (p_1' \wedge p_2) \vee (p_1' \wedge p_2')$. Furthermore, just as there are also Italians, Spaniards, Poles, Danes, Swedes, etc. in the EU each with their own national perspectives on Europeans, Form IV can also be extended to any number of propositions.

Form IV will be illustrated in the next section with two propositions, but the same rules of composition apply as the three preceding forms and the elementary grouping involving the affirmation and negation of a single proposition. Piaget thus concluded:

> There is nothing more, in fact, in these four forms than the progressive extension of the same operations $(\vee p)$ and $(\wedge p)$ hence one derives $(\wedge \bar{p})$ and $(\vee \bar{p})$: The Form II is the correlative[8] of Form I which itself extends directly [the elementary]

[8] According to Piaget (Piaget, Grize, 1972, pp. 256–257), the correlative and reciprocal operations can be derived from the inverse operation. $\bar{p} \wedge \bar{q}$ is the inverse of $p \vee q$, for example, and the operation involves two substitutions: conjunctions for disjunctions and vice versa, and affirmations for negations and vice versa. The outcome of performing just

grouping […]. Form III introduces two reciprocal implications there where Form I only knows one, and Form IV unites in a single whole all the operations developed in the preceding forms. There is thus only one grouping in four distinct forms, since the inverses, reciprocals and correlatives $(\vee p)$; $(\wedge \bar{p})$; $(\wedge p)$ and $(\vee \bar{p})$ are composable with each other. (Piaget, Grize, 1972, p. 333, my translation)

4.3. The Grouping of Binary Operators

Given a whole T that is partitioned dichotomously in two different ways by propositions p and q—$T = p \vee \bar{p}$ and $T = q \vee \bar{q}$, respectively—Form IV unites disjunctively the four parts engendered into a whole $T = (p \wedge q) \vee (p \wedge \bar{q}) \vee (\bar{p} \wedge q) \vee (\bar{p} \wedge \bar{q}) = (p * q)$. Although compound, the conjunctions are nevertheless propositions like any other; they can therefore be substituted for the propositions in Form I, and the substitutions constitute a grouping as follows (Piaget, Grize, 1972, Section 39 C):

Table 1

16 Distinct Logical Operators of Propositional Logic

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
pq	-	pq	-	-	pq	pq	-	pq	-	pq	-	pq	-	pq	-
$p\bar{q}$	-	$p\bar{q}$	-	$p\bar{q}$	-	-	$p\bar{q}$	$p\bar{q}$	-	-	$p\bar{q}$	$p\bar{q}$	-	-	$p\bar{q}$
$\bar{p}q$	-	$\bar{p}q$	-	$\bar{p}q$	-	$\bar{p}q$	-	-	$\bar{p}q$	-	$\bar{p}q$	-	$\bar{p}q$	$\bar{p}q$	-
$\bar{p}\bar{q}$	-	-	$\bar{p}\bar{q}$	$\bar{p}\bar{q}$	-	$\bar{p}\bar{q}$	-	$\bar{p}\bar{q}$	-	$\bar{p}\bar{q}$	-	-	$\bar{p}\bar{q}$	-	$\bar{p}\bar{q}$
$p * q$	o	$p \vee q$	$\bar{p} \wedge \bar{q}$	$p \mid q$	$p \wedge q$	$p \supset q$	$\overline{p \supset q}$	$q \supset p$	$\overline{q \supset p}$	$p \equiv q$	$p\ w\ q$	$p[q]$	$\overline{p[q]}$	$q[p]$	$\overline{q[p]}$

Note. The columns of this table are comprised of true conjunctions only (∧ is omitted to save space), and they are set out in pairs constituting the full complement of 4 conjunctions. Connecting the conjunctions in each column disjunctively generates the disjunctive normal forms of the logical operators in the bottom row. Except for *, *w*, $p[q]$, and $q[p]$ the binary operators are familiar. * denotes the complete affirmation; *w*, exclusive disjunction, and $p[q]$ as well as $q[p]$ are affirmations of p and q, in conjunction with either q and $\bar{q}$ or p and $\bar{p}$, respectively (based on Table 100 in Piaget, Grize, 1972, p. 214).

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the first substitution is the correlative $p \wedge q$; performing just the second operation, on the other hand, results in the reciprocal $\bar{p} \vee \bar{q}$. According to Halmos and Givant (1998, pp. 46–47), these operations are called "complement", "dual", and "contradual", respectively, and they depend on the principle of duality in a Boolean algebra. Moreover, these operations form a Klein four-group.

The logical operators of propositional logic can be expressed disjunctive normally as disjunctions of the conjunctions of Form IV. Via these disjunctive normal forms, Piaget shows that there are in fact 16 distinct binary operators (see Table 1). The columns of Table 1 are organised in complementary pairs with respect to the full complement of conjunctions, and, if the complementary pairs are composed disjunctively or conjunctively, the outcome is the complete affirmation or complete negation, respectively. These are the pendants to the laws of thought already highlighted in the elementary grouping (see Section 4.1), namely excluded middle and non-contradiction, respectively.

Table 1 sets out all 16 distinct logical operators, but Form IV is not simply a static taxonomy of logical operators. The interpropositional grouping is a system of transformations, and the logical operators can be transformed into each other as follows. Beginning with the equivalence $(p \wedge q) \vee (p \wedge \bar{q}) \vee (\bar{p} \wedge q) \vee (\bar{p} \wedge \bar{q}) = T$, e.g., the outcome of conjunctively composing the negation of the last conjunction with both sides of the equivalence is $[(p \wedge q) \vee (p \wedge \bar{q}) \vee (\bar{p} \wedge q) \vee (\bar{p} \wedge \bar{q})] \wedge \overline{(\bar{p} \wedge \bar{q})} = T \wedge \overline{(\bar{p} \wedge \bar{q})}$, i.e., $(p \wedge q) \vee (p \wedge \bar{q}) \vee (\bar{p} \wedge q) = \overline{(\bar{p} \wedge \bar{q})}$, which is equivalent to $(p \vee q)$ since $(p \vee q) = \overline{(\bar{p} \wedge \bar{q})}$. Algebraically, the transformation amounts to negating conjunctions of the complete affirmation and moving them to the opposite side of the equivalence, where they are composed conjunctively with the complete affirmation; for example, $(p \wedge q) \vee (\bar{p} \wedge q) \vee (\bar{p} \wedge \bar{q}) = T \wedge \overline{(p \wedge \bar{q})}$; i.e., $p \supset q = \overline{(p \wedge \bar{q})}$ (see Footnote 6b). By reversing the process, the original operator can then be restored. Moreover, $(p|q) \wedge \overline{(\bar{p} \wedge \bar{q})} = (p\,w\,q)$, for example, and $(p\,w\,q) \vee (\bar{p} \wedge \bar{q}) = (p|q)$. In other words, incompatibility $(p|q)$ is the outcome of composing reciprocal exclusion $(p\,w\,q)$ disjunctively with joint negation $(\bar{p} \wedge \bar{q})$. Whereas, the common part of an incompatibility $(p|q)$ and a disjunction $(p \vee q)$ $(= \overline{(\bar{p} \wedge \bar{q})})$ is $(p \wedge \bar{q}) \vee (\bar{p} \wedge q) = (p\,w\,q)$, a reciprocal exclusion. In short, the logical operators transform into each other, and the laws governing the system of transformations are those of a grouping. However, the conjunctions $\vee(p \wedge q)$ and $\wedge\overline{(p \wedge q)}$ rather than $\vee p$ and $\wedge\bar{p}$ of Form I constitute the direct and inverse operations of this manifestation of the grouping. And, the 16 logical operators defined disjunctive normally in Table 1 can be regarded as the operands of the grouping (cf. Seltman, Seltman, 1985).

The operations of the interpropositional grouping are as follows:

(i) The direct operation composes combinations of the four conjunctions constituting T disjunctively ($\vee$); e.g., $(o) \vee (p \wedge q)$; $(p \wedge q) \vee (p \wedge \bar{q})$.

(ii) The inverse operation is the negation of combinations of these conjunctions composed conjunctively ($\wedge$); e.g., $\wedge\overline{(p \wedge q)}$; $\wedge\overline{(p \wedge \bar{q})}$.

(iii) The general identity operation $\vee(o)$ leaves the elements it is composed with unaltered, e.g., $(p \wedge q) \vee (o) = (p \wedge q)$, and it is the product of the direct and inverse operations; e.g., $(p \wedge q) \wedge \overline{(p \wedge q)} = o$.

(iv) The special identities are:

a) Tautology: $(p \wedge q) \vee (p \wedge q) = (p \wedge q)$

b) Reabsorption: $(p \wedge q) \vee [(p \wedge q) \vee (p \wedge \bar{q})] = [(p \wedge q) \vee (p \wedge \bar{q})]$

c) Absorption : $(p \wedge q) \wedge (p * q) = (p \wedge q)$

(v) Associativity is again limited by the special identities.

In summary, this form of grouping engenders 16 distinct logical operators and unites them into a closed system of transformations. The interpropositional grouping thus represents operational transformations of a calculus of propositions, and, like the elementary grouping, the laws of thought are inherent in them; however, the transformations of the logical operators do not necessarily coincide with deductive inferences.

5. Implication, Transitivity, and Deduction

Via the direct operation of the elementary grouping, propositions p and $\bar{p}$ are composed disjunctively into a whole $p \vee \bar{p} = T$. The whole T is thus a proposition comprised of common $p \wedge T$ and non-common parts $\bar{p} \wedge T$ of p with T, i.e., $(p \wedge T) \vee (\bar{p} \wedge T) = T$. The fundamental operation of the elementary grouping thus engenders inclusions of parts in wholes. The conditional $p \supset T = (p \wedge T) \vee (\bar{p} \wedge T) \vee (\bar{p} \wedge \bar{T})$ is one of the few distinct logical operators already present in the elementary grouping. Since $\bar{T} = o$ therefore $\bar{p} \wedge \bar{T} = o$, $p \supset T$ converges with the affirmation $T[p] = (p \wedge T) \vee (\bar{p} \wedge T)$; the implications $p \supset T$, $\bar{p} \supset T$, $(p \vee \bar{p}) \supset T$, and $T \supset (p \vee \bar{p})$ are therefore expressions of the part-whole relations engendered by the fundamental operation of the interpropositional grouping. More generally, composing any two propositions x and y to form a whole z, $(x \vee y) = z$, via the direct operation of the interpropositional grouping generates relations of parts to the whole, which the following implications $x \supset z$; $y \supset z$; $(x \vee y) \supset z$ and $z \supset (x \vee y)$ express (Piaget, Grize, 1972, p. 343).

At this juncture, an ambiguity in Piaget's use of the term "implication" needs to be highlighted. In accordance with convention, Piaget uses "implication" and "conditional" synonymously to denote the logical operator. However, he also uses "implication" to denote the part-whole relations between propositions generated by the compositions of the interpropositional grouping. In such implications, the antecedent and consequent are related in some way. As a logical operator, implication $p \supset q$ is defined by $p \wedge q$, $\bar{p} \wedge q$, and $\bar{p} \wedge \bar{q}$ being true whilst $p \wedge \bar{q}$ is false. In part-whole relations on the other hand the truth of $p \wedge \bar{q}$ is excluded due to some relationship existing between the antecedent and consequent. For example, let p represent "$x \in$ mammals" and q, "$x \in$ vertebrates"; thus, some animals are mammalian vertebrates $p \wedge q$; some, non-mammalian vertebrates $\bar{p} \wedge q$; and others, neither mammalian nor vertebrate $\bar{p} \wedge \bar{q}$; however, invertebrates cannot be mammalian, so $p \wedge \bar{q}$ cannot be the case.[9] In this example, the antecedent and consequent are clearly related via their predicates, and Piaget (Piaget, Grize, 1972, pp. 226–227) distinguished implications referring to relations from implication as an operator and symbolised the former $p \longrightarrow q$.

According to Piaget (Piaget, Grize, 1972, p. 344), the primacy of implication is due to the transitivity of the nesting propositions it constitutes. But, first, referring to logical operators in general, surprisingly few are transitive. Piaget (Piaget, Grize, 1972, p. 340) illustrates intransitivity with mutual exclusions as follows:

$$(p|q) \wedge (q|r) \neq (p|r)$$
$$(p|q) = (\bar{p} \wedge \bar{q}) \vee (p \wedge \bar{q}) \vee (\bar{p} \wedge q)$$
$$(q|r) = (\bar{q} \wedge \bar{r}) \vee (q \wedge \bar{r}) \vee (\bar{q} \wedge r)$$
$$(p|q) \wedge (q|r) = (\bar{p} \wedge \bar{q} \wedge \bar{r}) \vee (p \wedge \bar{q} \wedge \bar{r}) \vee (\bar{p} \wedge q \wedge \bar{r}) \vee (\bar{p} \wedge \bar{q} \wedge r) \vee (p \wedge \bar{q} \wedge r)$$

Alternatively, it can be written in its dual form:

$$(p|q) \wedge (q|r) = (p * q * r) \wedge \overline{(p \wedge q \wedge r)} \wedge \overline{(p \wedge q \wedge \bar{r})} \wedge \overline{(\bar{p} \wedge q \wedge r)}$$

For example, let p = “$x \in$ invertebrate”, q = “$x \in$ vertebrate”, and r = “x lives attached to rocks (oysters, seaweed, etc.)”. The five triple conjunctions $(\bar{p} \wedge \bar{q} \wedge \bar{r})$ = neither invertebrate, nor vertebrate, nor living attached to rocks; $(p \wedge \bar{q} \wedge \bar{r})$; etc. are all possible; in fact, only invertebrate vertebrates attached to $(p \wedge q \wedge r)$ or not attached to rocks $(p \wedge q \wedge \bar{r})$, and non-invertebrate vertebrates attached to rocks $(\bar{p} \wedge q \wedge r)$ are not possible. Moreover, it is clear that the incompatibility $p|r$ does not hold since there are some invertebrates that live attached to rocks $(p \wedge \bar{q} \wedge r)$. Several of the triple conjunctions are thus true due to $(p|q)$ and $(q|r)$, and, they are also true in $(p|r)$; however, $(p|r)$ does not necessarily follow from $(p|q)$ and $(q|r)$ since $(p \wedge \bar{q} \wedge r)$ is one of the triplets that is compatible with both $(p|q)$ and $(q|r)$ but not with $(p|r)$. For transitive logical operators, on the other hand, the conclusion would hold for all of the conjunctions compatible with the premisses. According to Piaget (Piaget, Grize, 1972, p. 345), *conclusive deductions* are based on the transitive logical operators.

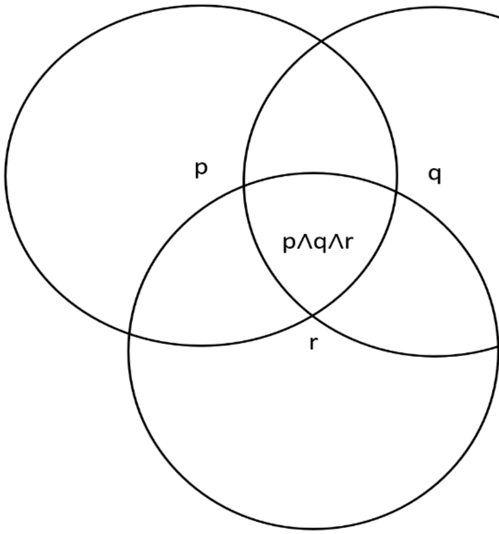
According to Piaget (Piaget, Grize, 1972, p. 344), conjunctions, implications, and equivalences, which are reciprocal implications, are the only transitive logical operators. In the case of transitivity of conjunctions $(p \wedge q) \wedge (q \wedge r) = (p \wedge r)$, let “$x \in$ both vertebrate and aquatic” $(p \wedge q)$ and “$x \in$ both aquatic and pulmonary” $(q \wedge r)$, for example, then, since whales, dolphins, etc. are pulmonary aquatic vertebrates, $(p \wedge r)$ is clearly true. However, $(p \wedge r)$ denotes all manner of pulmonary vertebrates, while $(p \wedge q \wedge r)$ only represents the small portion of them

[9] NB The conjunction $p \wedge q$ being true does not preclude the conjunctions $\bar{p} \wedge q$ and $\bar{p} \wedge \bar{q}$ also being true, although they are incompatible in classical logic (Apostel, 1982, Section 4).

inhabiting water. The transitivity of conjunctions is therefore founded on the three propositions p, q, and r having something in common (see Figure 4).

Figure 4

Transitivity of Conjunctions



Note. The intersections of three propositions p, q, and r are $(p \land q)$, $(q \land r)$, $(p \land r)$, and $(p \land q \land r)$ (based on Piaget, Grize, 1972, Fig. 51).

The operations of a grouping compose propositions, and Figure 4 represents a composition in which three propositions, p, q, and r, united into a whole $p \lor q \lor r$ are barely related. Nevertheless, they still have some common ground, and the transitivity of conjunctions bears on the minimum they still have in common: "the complete conjunction $(p \land q \land r)$ [which] is the lower bound (the greatest of the lower bounds) of p, q and r" (Piaget, Grize, 1972, p. 344). If, however, the propositions p, q, and r composing $p \lor q \lor r$ are related $p \supset q$ and $q \supset r$, then the clover-leaf shape of $p \lor q \lor r$ in Figure 4 takes on the form of a nesting hierarchy of inclusions like Figure 1. Transitivity is again due to what the propositions p, q, and r have in common; however, the common ground has now reached a maximum since the propositions are included in each other. This is the smallest of the unions three distinct propositions form, and, according to Piaget (Piaget, Grize, 1972, p. 344), it is an upper bound on transitivity. For equivalences $p \equiv q$, $q \equiv r$, then $p \equiv r$, on the other hand, the upper and lower bounds of the transitivity of propositions p, q and r coincide since $p \equiv q \equiv r$ (Piaget, Grize, 1972, p. 344). In short, transitivity is founded on what propositions have in common, and conjunctions, implications, and equivalences are the only transitive logical operators of the grouping, and implication marks the upper boundary of transitivity.

While transitivity is key to conclusive deductions, it is nevertheless quite rare among the logical operators of the interpropositional grouping. The operatory perspective on deduction might therefore appear to be inconsistent with actual deductive reasoning. However, the grouping unites logical operators into a closed system of transformations, and, via its operations, other operators interact with the few transitive operators. Given p, q, r, and $p \supset q$, for example, operators such as $(p \wedge r)$, $(q \wedge r)$; $(p \vee r)$, $(q \vee r)$; $(q|r)$; $(p|r)$; etc. are able to participate in the transitivity of implications; $(p \wedge r) \supset (q \wedge r)$; $(p \vee r) \supset (q \vee r)$; $(q|r) \supset (p|r)$; etc. therefore hold if $p \supset q$ holds. A richness of deductions commensurate with that of deductive reasoning is therefore generated by the many non-transitive operators participating in the transitivity of the few.

In summary, the operations of the interpropositional grouping compose propositions with one another, and some compositions engender part-whole relations between propositions. Implications as expressions of these part-whole relations thus go hand-in-hand with the fundamental operations of the interpropositional grouping. The conditional operator is one of the few operators already present in the elementary grouping, and the Forms I–IV of implication systematically extend the elementary grouping to multiple propositions by progressively differentiating the part-whole relations imminent in its propositions. These Forms thus propagate part-whole relations between propositions and thereby proliferate implications in the sense of relations. Moreover, transitivity is based on the nesting propositions engendered by the interpropositional grouping, and implication is not only one of the few transitive logical operators but also represents an upper bound on transitivity. Along with equivalence, which is in fact a double implication, implication is thus the primary source of conclusive deductions. In short, implication plays a fundamental role in the interpropositional grouping, and, along with equivalence, it accounts for the deductive fertility of this grouping (Piaget, Grize, 1972, p. 346).

6. The Nature of Deduction According to Piaget

The previous section has shown how the interpropositional grouping makes deduction possible. The nature of deduction is therefore tied up with the nature of the interpropositional grouping, and, in this section, I will attempt to shed light on the nature of deduction indirectly by characterizing the nature of the interpropositional grouping.

According to Piaget, the interpropositional grouping has synchronic and diachronic aspects. Starting with the latter, intelligence is a natural continuation of the biological adaptation of organisms (e.g., Piaget, 1952, Chapter Introduction; 1971b; 2001, Chapter 1). Organisms are open, self-regulating systems; as such, they are existentially dependent on their environments, and they strive to strike a balance between the demands of the environment on the one hand and the integrity of their biological organisations on the other through self-regulation. Like the biological organism, intelligence also has an internal organisation, adapts to its environment, and strives toward equilibrium; unlike the biological organism, though, intelligence actually achieves states of equilibrium.

Moreover, intelligence evolves in a sequence of stages over time (e.g., Piaget, 1977; 2001), and the sensorimotor, semiotic, concrete-operational, and formal-operational are the widely accepted stages, although their number varies in Piaget's works (Kesselring, 2009). These stages can be more broadly categorised into pre-operational—the sensorimotor, semiotic—and operational—concrete and formal—stages. As the terminology suggests, intra- and interpropositional operations occur at the operational stages.

The first cognitive equilibria are achieved at the operational stages, but they are presaged by coordinations of voluntary actions involving sensory stimuli and motor responses during the sensorimotor stage. The advent of language at the semiotic stage then heralds a change. The physical world constructed at the sensorimotor stage gradually becomes immersed in a world of representations. The effects of this immersion are twofold: on the one hand, the representational world not only captures the physical reality constructed at the sensorimotor stage but transcends it in all directions; on the other hand, the manipulations of objects, still enactive at the sensorimotor stage, can now be performed solely in the mind without physical manipulation accompanying them. The latter development is interiorization, and a whole new level of interiorization is achieved with the advent of operations (Piaget, 2001; Piaget, Grize, 1972, pp. 14–15).

Operations are interiorised actions, and just as actions occur in coordination with other actions, operations occur in concert with other operations. According to Piaget, equilibrium is achieved, however, when these operations are coordinated with others to form systems of transformations that are completely reversible. With the emergence of equilibria, the diachronic aspect is complemented by a synchronic aspect.

Turning to the synchronic aspect, operations in states of equilibrium form structured wholes amenable to formalisation, and psycho-logic models them using algebraic tools of logic. Groupings are thus formalisations of the structured wholes intra- and interpropositional operations form in states of equilibrium; as such they are new constructions, but they have functional roots in fundamental biological mechanisms (Piaget, Grize, 1972, pp. 14–15).

The biological roots come to expression in the cognitive function of the interpropositional grouping. Given two observable phenomena represented by propositions p and q, it is not immediately obvious how they are related to each other. Conjunctions $p \wedge q$, $\overline{p} \wedge q$, $p \wedge \overline{q}$, and $\overline{p} \wedge \overline{q}$ represent the four possible ways the phenomena can be associated in observation; however, individually each observation does not allow the relationship between the phenomena to be determined. Observation of p and q always occurring together, $p \wedge q$, for example, could mean that p and q are related in any of the 8 ways represented by the columns in Table 1 in which $p \wedge q$ occurs. Through observation of the occurrences of the four possible associations of p and q, on the other hand, the exact relationship between the phenomena can be determined. Observation of $p \wedge q$ and $\overline{p} \wedge \overline{q}$ occurring without exception but no cases of either $\overline{p} \wedge q$ or $p \wedge \overline{q}$, for example, indicates that the phenomena represented by p and q are equivalent; whereas observation of $p \wedge q$, $\overline{p} \wedge q$, and $\overline{p} \wedge \overline{q}$ but no cases of $p \wedge \overline{q}$ means that p implies q (see Table 1). The interpropositional grouping thus serves as a cognitive tool for determining connections between observable phenomena and therefore represents a cognitive adaptation to the environment.

According to Piaget, three key ideas characterise structures: "the idea of wholeness, the idea of transformation, and the idea of self-regulation" (Piaget, 1970, p. 5). Piaget draws attention to the relational nature of parts and whole in structures; however, the whole is neither the sum of its parts nor are the parts wholly determined by the whole. Neither the whole nor the parts are primary, and, instead of bottom-up or top-down constructions, the parts and whole are the outcome of laws of construction that are both structured and structuring. Moreover, the parts are transformed by the system's laws of composition, but the system of transformations as a whole is closed since the outcomes of these transformations also belong to the system and preserve its laws. In the interpropositional grouping of logical operators, for example, neither the operands, the 16 logical operators, nor the whole structure, the grouping, are primary; they are the outcome of interpropositional operations of thought achieving a state of equilibrium. Moreover, the operations of the grouping are laws of composition that transform the logical operators operated on, and the outcome of these operations is another logical operator that also preserves the laws of the system; the system of operations formed by the grouping is, therefore, closed and self-regulating. Like its namesake the group, the grouping of interpropositional operations thus fulfils Piaget's characterisation of a structure.

Since Frege, it has been standard practice to axiomatize logic in analogy with the substantial axiomatisations of extra-logical sciences (Hintikka, Sandu, 2007, Section 5). Accordingly, logic is reduced to a handful of axioms and rules of inference, from which all the formulae of logic can be derived. Axiomatisations of logic like those of extra-logical sciences are therefore systematisations of a theory; nonetheless, there are significant differences between the two. Substantial theories are sets of propositions that correspond to an extra-logical reality, and, by reducing these propositions to a handful of postulates from which those describing or predicting the targeted realities can be derived, axiomatisations assist in understanding these realities. Moreover, in the axiomatisation of substantial theories, the derivations correspond to what is ordinarily understood by deduction, and the correspondence of the theory with reality as well as verification of its predictions tend to transmit truth backwards to the axioms. Like substantial axiomatisations, the axioms of a formal system are the underived formulae on which the derivation of other formulae of the theory are founded. However, there is no difference in principle between the derived formulae and the axioms of a formal system—the latter are simply formulae without premisses.

Moreover, any set of formulae can serve as axioms as long as they are consistent—a formula and its contradictory cannot be derived from them—preferably independent, and semantically complete—all the true formulae of the theory can be derived from them. Moreover, in contrast to a substantial axiomatisation, the derivation of the formulae in formal systems need not correspond to the rules of inference in logic; a mechanical means of systematically listing all of the formulae is sufficient. Despite similarities with formal systems, the interpropositional grouping models an extra-logical reality; it is therefore a substantial axiomatisation, specifically a substantial axiomatisation of the equilibrium achieved by interpropositional operations of thought.

The interpropositional grouping models the reversibility of rational thought, on the one hand, and the systems of transformations operations of thought engender when they achieve equilibrium, on the other. According to Piaget (Piaget, Grize, 1972, Sections 36–38), the interpropositional grouping represents either a relaxation of the strict reversibility of groups through augmentation with inclusions and self-inclusions or a tightening of the operations of a lattice through the introduction of reversibility into its operations. Although not purely abstract, the interpropositional grouping is thus a mathematical structure that lies mid-way between groups and lattices. According to Grize (2013, p. 152), Piaget based the interpropositional grouping on Boolean structures.

There are many expressions for each of the different logical operators; despite their disparate guises, though, they can be shown to be equivalent by reducing them to their normal forms. In fact, Table 1 represents a classification of equivalent formulae via their disjunctive normal forms. $\{pq \vee \bar{p}q \vee p\bar{q}\}$ thus represents the class of formulae $p \vee q$, $p \vee q \vee q$, etc; $\{T\}$, the class of tautologies $p \vee \bar{p}$, $q \vee \bar{q}$, etc., and the class of contradictions $p \wedge \bar{p}$, $q \wedge \bar{q}$, etc., being empty, is represented by the null class $\{o\}$. Moreover, the propositional connectives $\vee$, $\wedge$, and $\bar{\ }$ are congruent with operations $\cup$, $\cap$, and ’, respectively, on these classes (Rutherford, 1966, pp. 50–51). The operations of this grouping, therefore, correspond to operations on classes of the classification of formulae; transforming $p\,w\,q$ into $p \vee q$ via the direct operation $\vee pq$, for example, is $(p\,w\,q) \vee pq = (\bar{p}q \vee p\bar{q}) \vee pq = p \vee q$, which corresponds to $\{\bar{p}q \vee p\bar{q}\} \cup \{pq\} = \{\bar{p}q \vee p\bar{q} \vee pq\}$ in terms of classes; transforming $p \vee q$ back to $p\,w\,q$ via the inverse operation $\wedge \overline{(pq)}$, on the other hand, corresponds to the relative complement of $\{pq\}$ in $\{p \vee q\} = \{\bar{p}q \vee p\bar{q} \vee pq\}$, i.e., $\{\bar{p}q \vee p\bar{q}\} = \{p\,w\,q\}$. The general identity is composed of the direct and inverse operations of a grouping, and it leaves any element of the grouping unaltered $r \vee o = r$ but $r \wedge o = o$; in terms of the classification of formulae, $\{r\} \cup \{o\} = \{r\}$ and $\{r\} \cap \{o\} = \{o\}$; hence $\{o\} \leqq \{r\}$ for all $\{r\}$. T, on the other hand, is a special identity $T \vee r = T$ and $T \wedge r = r$; therefore $\{T\} \cup \{r\} = \{T\}$ and $\{T\} \cap \{r\} = \{r\}$; hence $\{r\} \leqq \{T\}$ for all $\{r\}$. In other words, all the classes in the classification of formulae include $\{o\}$ and are included in $\{T\}$; the classification therefore has a null $\{o\}$ and a universal element $\{T\}$. Moreover, each element r of the grouping has an inverse such that $r \wedge \bar{r} = o$ and $r \vee \bar{r} = T$; for every $\{r\}$ there is therefore a $\{s\}$ such that $\{r\} \cap \{s\} = \{o\}$ and $\{r\} \cup \{s\} = \{T\}$, i.e., a complement.

Since the operations of the grouping are also distributive, this complement is unique and can be denoted *r*’. The equivalence classes of formulae under the lattice operations corresponding to the operations of the grouping, therefore, constitute a complemented distributive lattice. Since a Boolean algebra is a complemented distributive lattice (Halmos, Givant, 1998; Rutherford, 1966), the structure Piaget loosely characterised as being mid-way between groups and lattices seems to correspond to a Boolean algebra.

Halmos (2016, Chapter Introduction Section 2) remarked that “Boolean algebras have an almost embarrassingly rich structure [...]. In every Boolean algebra there is, moreover, a natural order relation [and] [t]he algebraic structure and the order structure are as compatible as they can be”. It is therefore desirable to narrow down the nature of deduction still further.

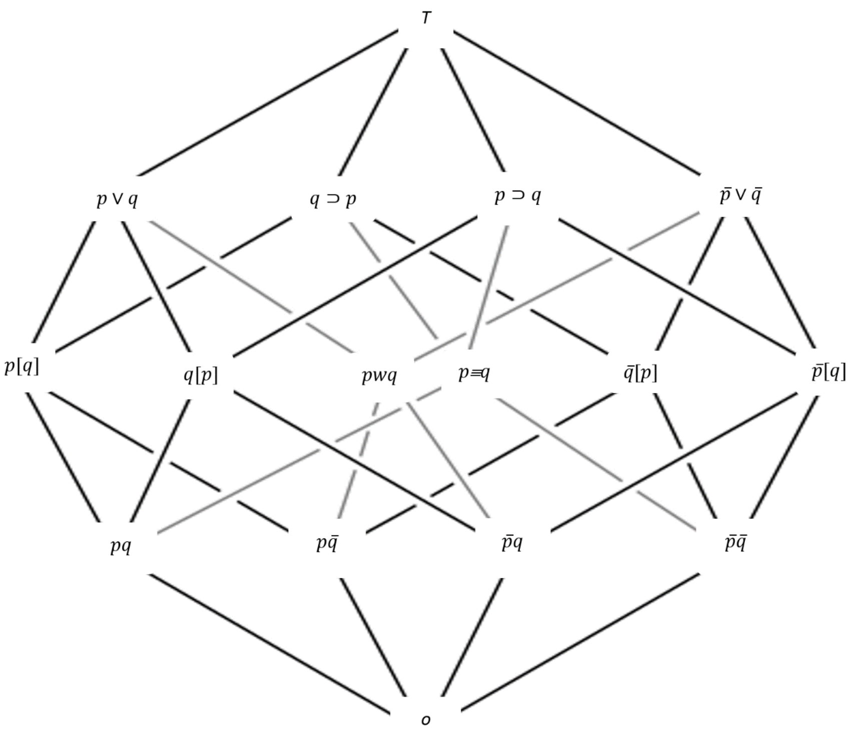
From the viewpoint of classical logic, $p \supset q$ has the same truth conditions as $\bar{p} \vee q$, namely, true except when p is true and q is false. Since arbitrary propositions may be substituted into the propositional variables p and q, it is not possible to preclude the falsity of the compound proposition without imposing some additional constraints. In a free Boolean algebra, the postulates constitute the only constraints on propositions. To determine the additional constraints on p and q necessary for $p \supset q$ to be true without exception, consider any two propositions p and q belonging to the classification of propositional formulae in Table 1 and the class $\{p\}' \cup \{q\}$, which corresponds to composing q disjunctively with the negation of p, $\bar{p} \vee q$. Clearly $p \supset q$ is also a member of the class $\{p' \cup q\}$, but, for it to be true without exception, $\{p\}' \cup \{q\} = \{T\}$, i.e., $\{q\} \geqq \{p\}$ so that $\{q\}$ is in the interval $[\{p\}, \{T\}]$ (Rutherford, 1966, pp. 51–52).

Figure 5 (on the next page) is a Hasse diagram of the equivalence classes in Table 1. Although it is simply an alternative representation, it has the advantage of bringing the lattice structure clearly to the fore. Referring to Figure 5, the condition set out in the previous paragraph is fulfilled provided $\{q\}$ is a class of propositions occupying a node on one of the lines connecting $\{p\}$ with $\{T\}$; for example, $p[q]$, $q[p]$, $p \vee q$, etc., are propositions belonging to classes on the line connecting the class $\{p \wedge q\}$ with $\{T\}$; the implications $p \wedge q \supset p[q]$, $p \wedge q \supset q[p]$, $p \wedge q \supset p \vee q$, etc. are therefore tautologies. From the viewpoint of Table 1, $p \wedge q$ implies all those logical operators in which it is affirmed in the disjunctive normal form.

In Section 5, I mentioned how Piaget distinguished between implications as relations and as operators. In essence, relations in contrast to operators cannot be false due to some relation existing between the propositions, and I illustrated the difference with an implication in which the antecedent and consequent are related via their predicates. By means of the lattice structure, it is possible to deal with such relations more generally. On the one hand, if $\{q\}$ is in the interval $[\{p\}, \{T\}]$, $p \supset q$ is a tautology, and, in Piaget’s terminology, it is an implication in the relational sense. Moreover, the order relation between the classes is $\{q\} \geqq \{p\}$, which is also known as an inclusion relation since it is equivalent to

Figure 5

Hasse Diagram of the 16 Logical Operators of Propositional Logic



Note. The figure represents the projection onto the plane of a four-dimensional cube. The logical operators occupy the points of intersecting lines, and lines connecting points represent inclusion relations. Thus $p \supset q \geqq p \equiv q$, $q[p]$, $\bar{p}[q]$, pq, $\bar{p}q$, $\overline{pq}$ and o; but not $p\bar{q}$ (after Rutherford, 1966, Fig. 7).

$\{p\} = \{p\} \cap \{q\} \equiv \{p\} \cup \{q\} = \{q\}$. The same inclusion relation can also be expressed, admittedly less conventionally, in terms of parts and wholes. In Piaget's parlance, then, $\{p\}$ being a part of the whole $\{q\}$, i.e., $\{p\} = \{p\} \cap \{q\}$ or equivalently $\{p\} \cup \{q\} = \{q\}$, thus refers to an implication $p \rightarrow q$ that cannot be false because an inclusion relation $\{q\} \geqq \{p\}$ exists between the antecedent and consequent. Moreover, by generalising the elementary interpropositional grouping formed by the affirmation and negation of a single proposition p to multiple propositions, Piaget, in effect, inserted propositions q, r, s, etc. in the interval $[\{p\}, \{T\}]$ of the elementary grouping. The implications $p \supset q$, $q \supset r$, etc. engendering the forms of implication are thus implications in the sense of relations $p \rightarrow q$, $q \rightarrow r$, etc., and Piaget's allusions to part-whole relations in describing these implications seem in fact to correspond to inclusion relations (Winstanley, 2021, Section 3.1).

Piaget discerned four different forms of implication, but only Forms I–III give rise to conclusive deductions. Developmentally, the interpropositional grouping synthesises intrapropositional groupings of operations on relations and classes into a single structure, and Forms I and III of implication can be modelled by operations on classes, whereas operations on relations model Form II. Moreover, the part-whole relations between propositions are the basis for deduction in Forms I and III; deduction in Form II on the other hand is based on the transitivity of the order relation. Lattices have two equivalent definitions (see Footnote 3), one emphasising operations; the other, being based on a poset, highlighting their relational nature. Moreover, they are connected by the identity $y = x \cap y \equiv x \geqq y \equiv x \cup y = x$. Order and inclusion relations, two seminal characteristics of lattices, are therefore inherent in the Forms I–III of implication. Piaget thus appears to have attributed the nature of deduction specifically to the lattice structure inherent in the embarrassing richness of a Boolean algebra.

Moreover, Piaget attributed the deductive richness of reasoning to propositions participating in the transitivity of logical operators like implication via the operations of the grouping. With the help of lattice theory, this can be circumscribed precisely: "The totality of valid deductions from a proposition or set of axioms *p* are [...] those propositions belonging to the classes of the interval [[{*p*}, {*T*}]]" (Rutherford, 1966, p. 52). According to Piaget, From IV does not give rise to any new implications; however, as part of the algebraic rather than the order structure of a Boolean algebra, it can nevertheless contribute to the deductive richness of reasoning.

7. Conclusion

According to Piaget, the nature of human propositional reasoning lies in the interpropositional grouping, the calculus embodied in propositional operations, and the nature of propositional deduction, in particular, lies in the relations between propositions inherent in the Forms I–III of implication. If the interpropositional grouping constitutes a Boolean algebra, as I have argued, then the nature of deduction lies specifically in the order rather than the algebraic structures of this embarrassingly rich structure. I therefore conclude that the nature of deduction according to Piaget lies specifically in the lattice engendered by the operations of the interpropositional grouping.

Finally, having characterised the nature of deduction, it would be remiss not to touch at least briefly on its implications for logic. How Piaget regarded the relationship between the forms of implication and axiomatisations of propositional logic was touched on briefly at the end of Section 4.2.3. Put succinctly, the interpropositional grouping is the natural structure inherent in propositional reasoning, which "lies 'beneath' the operations codified by axioms [of logic]" and furnishes "the underpinnings of logic" (Piaget, 1970, p. 31). In other words, the interpropositional grouping forms the foundation for propositional logic. However, propositional logic is not synonymous with the interpropositional grouping. According to Piaget, "logic is the mirror of thought, and not vice versa" (Piaget, 2001, p. 27), and, after several iterations, Piaget eventually defined logic without the aid of metaphor as "the formal theory of deductive operations" (Piaget, Grize, 1972, p. 20, authors' emphasis). Piaget's psychological theory of propositional reasoning therefore forms an evidential basis for a logic conceived as a formal theory (Winstanley, 2021), and the forms of implication will clearly play a seminal role in its construction. To my knowledge, a Piagetian logic has yet to be constructed (Apostel, 1982; Grize, 2013); if it were, however, it would arguably constitute a natural logic among the plurality of logics since a logic is imminent in a structure (Shapiro, 2014), and the logic imminent in a natural structure like the interpropositional grouping would constitute a natural logic.

REFERENCES

Apostel, L. (1982). The Future of Piagetian Logic. *Revue Internationale de Philosophie*, *36*, 567–611.

Bond, T. G. (1978). Propositional Logic as a Model for Adolescent Intelligence—Additional Considerations. *Interchange*, *9*, 93–98. doi:10.1007/BF01816518

Bond, T. G. (2005). Piaget and Measurement II: Empirical Validation of the Piagetian Model. In L. Smith (Ed.), *Critical Readings on Piaget* (pp. 178–208). Abingdon: Routledge.

Byrne, R. M. J., Johnson-Laird, P. N. (2009). 'If' and the Problems of Conditional Reasoning. *Trends in Cognitive Sciences*, *13*(7), 282–287. doi:10.1016/j.tics.2009.04.003

George, R. (1997). Psychologism in Logic: Bacon to Bolzano. *Philosophy & Rhetoric*, *30*(3), 213–242.

Grize, J.-B. (2013). Operatory Logic. In B. Inhelder, D. de Caprona, A. Cornu-Wells (Eds.), *Piaget Today* (pp. 149–164). Abingdon: Taylor and Francis.

Halmos, P. R. (2016). *Algebraic Logic*. Mineola, NY: Dover Publications Inc.

Halmos, P. R., Givant, S. (1998). *Logic as Algebra* (Vol. 21). Retrieved from: http://archive.org/details/logicasalgebra0000halm

Harman, G. (1984). Logic and Reasoning. In H. Leblanc, E. Mendelson, A. Orenstein (Eds.), *Foundations: Logic, Language, and Mathematics* (pp. 107–127). Dordrecht: Springer Netherlands. doi:10.1007/978-94-017-1592-8_7

Harman, G. (1986). *Change in View: Principles of Reasoning*. Cambridge, MA: The MIT Press.

Hintikka, J., Sandu, G. (2007). What is Logic? In D. Jacquette (Ed.), *Philosophy of Logic* (pp. 13–40). Netherlands: North Holland.

Inhelder, B., Piaget, J. (1958). *The Growth of Logical Thinking from Childhood to Adolescence*. London: Routledge, Chapman and Hall.

Jacquette, D. (2007). Introduction: Philosophy of Logic Today. In D. Jacquette (Ed.), *Philosophy of Logic Today* (pp. 1–12). Netherlands: North Holland.

Johnson-Laird, P. N. (1999). Deductive Reasoning. *Annual Review of Psychology*, *50*(1), 109–135.

Johnson-Laird, P. N. (2006). *How we Reason*. Oxford: Oxford University Press.

Johnson-Laird, P. N., Byrne, R. M. J., Schaeken, W. (1992). Propositional Reasoning by Model. *Psychological Review*, *99*(3), 418–439.

Kesselring, T. (2009). The Mind's Staircase Revised. In U. Mueller, J. I. M. Carpendale, L. Smith (Eds.), *The Cambridge Companion to Piaget* (pp. 372–399). New York: Cambridge University Press.

Piaget, J. (1952). *The Origins of Intelligence in Children*. New York: International Universities Press Inc.

Piaget, J. (1957). *Logic and Psychology*. New York: Basic Books Inc.

Piaget, J. (1970). *Structuralism*. New York: Basic Books Inc.

Piaget, J. (1971a). Genetic Epistemology. New York: W. W. Norton & Company Inc.

Piaget, J. (1971b). *Biology and Knowledge: An Essay on the Relations Between Organic Regulations and Cognitive Processes*. Chicago; London: University of Chicago Press.

Piaget, J. (1977). The Stages of Intellectual Development in Childhood and Adolescence. In H. E. Guber, J. J. Vonèche (Eds.), *The Essential Piaget* (pp. 814–819). New York: Basic Books, Inc.

Piaget, J. (2001). *The Psychology of Intelligence*. London; New York: Routledge.

Piaget, J., Beth, E. W. (1966). *Mathematical Epistemology and Psychology* (vol. 12). Dordrecht, Holland: Springer Netherlands. doi:10.1007/978-94-017-2193-6

Piaget, J., Grize, J.-B. (1972). *Essai de logique opératoire* [Essay on Operational Logic] (vol. 15). Paris: Dunod.

Posy, C. J. (1997). Between Leibniz and Mill: Kant's Logic and the Rhetoric of Psychologism. *Philosophy & Rhetoric*, *30*(3), 243–270.

Restall, G., Beall, J. C. (2000). Logical Pluralism. *Australasian Journal of Philosophy*, *78*(4), 475–493. doi:10.1080/00048400012349751

Restall, G., Beall, J. C. (2001). Defending Logical Pluralism. In J. C. Beall, G. Restall (Eds.), *Proceedings of the 1999 Conference of the Society of Exact Philosophy* (pp. 1–22). Stanmore: Hermes Science Publishers.

Rips, L. J. (2008). Logical Approaches to Human Deductive Reasoning. In J. E. Adler, L. J. Rips (Eds.), *Reasoning. Studies of Human Inference and Its Foundations* (pp. 187–205). Cambridge: Cambridge University Press.

Russell, G. (2019). Logical Pluralism. In E. N. Zalta (Ed.), *The Stanford Encyclopedia of Philosophy* (Summer 2019). Retrieved from: https://plato.stanford.edu/archives/sum2019/entries/logical-pluralism/

Rutherford, D. E. (1966). Introduction to Lattice Theory. Edinburgh, London: Oliver and Boyd Ltd.

Seltman, M., Seltman, P. (1985). Piaget's Logic: A Critique of Genetic Epistemology. London: George Allen & Unwin.

Shapiro, S. (2014). Varieties of Logic. Oxford: OUP.

Smith, L. (1987). A Constructivist Interpretation of Formal Operations. *Human Development*, *30*(6), 341–354. doi:10.1159/000273192

Smith, L., Mueller, U., Carpendale, J. I. M. (2009). Introduction. Overview. In L. Smith, U. Mueller, J. I. M. Carpendale (Eds.), *The Cambridge Companion to Piaget* (pp. 1–44). New York: Cambridge University Press.

Stenning, K., van Lambalgen, M. (2008). *Human Reasoning and Cognitive Science*. Cambridge, MA, London: The MIT Press.

Stenning, K., van Lambalgen, M. (2011). Reasoning, Logic, and Psychology. *Wiley Interdisciplinary Reviews: Cognitive Science*, *2*(5), 555–567. doi:10.1002/wcs.134

van Benthem, J. (2007). Logic in Philosophy. In D. Jacquette (Ed.), *Philosophy of Logic* (pp. 65–99). Amsterdam: North-Holland. doi:10.1016/B978-044451541-4/50006-3

van Benthem, J. (2008). Logic and Reasoning: Do the Facts Matter? *Studia Logica*, *88*(1), 67–84. doi:10.1007/s11225-008-9101-1

Winstanley, M. A. (2021). A Psychological Theory of Reasoning as Logical Evidence: A Piagetian Perspective. *Synthese*. doi:10.1007/s11229-021-03237-x

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Article

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A PROBABILISTIC TRUTH-CONDITIONAL SEMANTICS FOR INDICATIVE CONDITIONALS[1]

SUMMARY: In my article, I present a new version of a probabilistic truth prescribing semantics for natural language indicative conditionals. The proposed truth conditions can be paraphrased as follows: an indicative conditional is true if the corresponding conditional probability is high and the antecedent is positively probabilistically relevant for the consequent or the probability of the antecedent of the conditional equals 0. In the paper, the truth conditions are defended and some of the logical properties of the proposed semantics are described.

KEYWORDS: indicative conditionals, conditional probability, connection intuition.

1. Introduction

In my article, I will present a new version of a probabilistic truth prescribing semantics for natural language indicative conditionals. In this introductory section, I will present the basic notions and ideas which will be helpful in the rest of the article. In the second section, I will present the natural predeces-

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[1] This paper is based on my master thesis defended at the University of Barcelona in 2015 under the supervision of Max Kölbel. I would like to thank Max Kölbel, José Martínez, Rafał Urbaniak, Jan Sprenger, Paweł Pawłowski, Vincenzo Crupi, Andrea Iacona, Naomi Coldenhoff, and the anonymous referees for their useful comments.

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sor of my theory—Adams' probabilistic theory of indicative conditionals. The third section will be devoted to the new version of the theory itself. In the fourth section, I will discuss the problem of compound conditionals (e.g., "If Martha is in the kitchen, we will have dinner soon, and if Marv is in the garage, the car will be fixed tonight"). The fifth and final section will discuss some of the implications of my theory.

1.1. Indicative Conditionals

Defining indicative conditionals is not an easy task. I will do it by defining counterfactual conditionals which constitute the complement of indicative conditionals in the set of all conditionals. Some examples of counterfactual conditionals are:

C1 If he had not tampered with the machine, it would not have broken down. [(165)a.]

C2 He would make more progress if he were using a computer. [(164)b.]

Typical counterfactual[2] conditionals share two features:

CM1 The counterfactual conditionals use "would" as the auxiliary of its main verb.

CM2 A speaker who uses a counterfactual conditional implies that the antecedent is false.

Conditionals that do not share one of these characteristics will be called indicative conditionals. From now on whenever I will write "conditional" I mean an indicative conditional.

Here are some examples:

(1) If it rains a lot, the ground will become waterlogged. [(77)a.][3]

(2) People burn (instead of tanning) if they have a white, freckled skin. [(85)c.]

(3) If you press this button, the fire alarm goes off. [(548)c.]

(4) If the witness is prepared to testify, we have a strong case against Harry Field. [(548)c.]

(5) If he does not say anything, he will not betray us. [(553)a.]

(6) If his daughter is beautiful, my daughter is a Venus! [(653)b.]

[2] For the summary of the discussion about the demarcation of conditionals, including examples of less typical counterfactuals, see the work of Bennett (2003), which is also the source of CM1 and CM2.

[3] Most of the examples that I use come from (Declerck, Reed, 2001). In the brackets, I will place the number of examples from the book. In case I modified the example I will mark it by adding * to the number in the bracket.

(7) People ignore the warning if you do not point out the consequences. [(919)a.]

(8) If you won all the fights, I am Cassius Clay. [(587)b.]

By theories of conditionals I mean theories that define the semantic value (truth or acceptability) of simple conditionals on the basis of some properties of its antecedent and consequent.[4] Probabilistic theories of conditionals define the semantic value of conditionals on the basis of the conditional probability of the consequent given the antecedent.

1.2. Connection Intuition

Almost everything about conditionals is controversial. At the same time, it seems that there is at least one widely shared intuition: every positively valued conditional involves the existence of a connection between antecedent and consequent. Before further elaboration, the intuition has to be restricted, for not all of the conditionals used by competent speakers involve such a link. For example, (8) does not involve any kind of connection. Still, conditionals such as (8), so-called Dutchman conditionals, are sufficiently rare and specific to not disqualify the intuition. I will call the main body of conditionals, which involve the connection, *canonic conditionals*. The Dutchman conditionals will be discussed in the third section.

Further elaboration of the notion of connection is a tricky task. If we define it too narrowly, some positively valued canonic conditionals will be left outside. If we define it too broadly, the connection thesis becomes a trivial one. More than that just a few examples (1)–(7) show that in some cases the connection is hard to specify. In cases like (1), (2), or (3) the categorization is quite straightforward: the connection is clearly causal.

On the other hand, in the case of (6), it is not quite clear what the nature of the link is. Still, there are ways to argue for the existence of such a connection even in such cases. For example, we can claim that every situation which would justify the utterance of such conditionals has to involve some link, such as the following:

> Two women, Jane and Alice, talk about the countenance of the daughter of their common friend, Susan. Jane claims that Susan is beautiful. On the basis of that statement, Alice reasons about Jane's aesthetic taste and her criterion of beauty. On the basis of both, she infers that Jane would also categorize her (Alice's) daughter as beautiful and claims (6).

[4] Historically the first theory of conditionals in that sense was the theory of the material implication. It identifies truth conditions of natural language conditionals with truth conditions of material implication. The theory suffers from many counterexamples and, despite many defense attempts, seems to be disqualified (for examples of the defense of the theory, see Ajdukiewicz, 1956; Grice, 1989; see Edgington, 1995 for criticism).

So even in the case of (6), there seems to be some, in this case inferential, connection between the antecedent and the consequent.

Perhaps despite the diversity of conditionals, some additional characterization of the connection common to all true canonical conditionals could be given. For example, the inferential theories of conditionals (e.g., Douven, Elqayam, Krzyżanowska, 2022; Krzyżanowska, Wenmackers, Douven, 2014) leverage the fact that the connections between antecedents and consequents of true conditionals can be unpacked in a form of valid arguments. For example, in the case of (6) the argument can resemble the inference made by Alice. This feature of the connection inherent to true conditionals was developed into the following truth conditions:

Definition 1. A speaker *S*'s utterance "If *p*, *q*" is true iff (i) *q* is a consequence—be it deductive, abductive, inductive, or mixed-of *p* in conjunction with *S*'s background knowledge, (ii) *q* is not a consequence—whether deductive, abductive, inductive, or mixed—of *S*'s background knowledge alone but not of *p* on its own, and (iii) *p* is deductively consistent with *S*'s background knowledge or *q* is a consequence (in the broad sense) of *p* alone (Krzyżanowska, Wenmackers, Douven, 2014, p. 5).

The theory is surely promising, on the other hand, Definition 1 contains many concepts meaning of which is still controversial. For example, it is unclear which logical system determines when a deductive argument in question is valid. The same goes for inductive and abductive arguments. These gaps, acknowledged in (Douven, Elqayam, Krzyżanowska, 2022), will undoubtedly be filled in the future, and these developments will likely lead to plausible, fully-fledged theories of conditionals. In this paper, I will develop a different way of conceptualizing the connection. I will use the fact that the link between antecedents and consequents of many conditionals is positive and probabilistic. By probabilistic I mean merely that we can capture that connection in probabilistic terms. By positive I mean that the occurrence of what is described by the antecedent makes more likely the occurrence of what is described by the consequent. It is easy to see that the links involved in our examples (stated in proper contexts) meet these two requirements.

I will show that a version of probabilistic semantics is able to capture the connection by exploiting these two properties. I present two versions of probabilistic semantics: in the next section, the classical theory by Adams, and in the third section a new proposal.

2. The Traditional Version of the Probabilistic Theory and Its Problems

By "traditional probabilistic theory of conditionals"[5] I mean the proposal developed by Ernest Adams in his (1975). It has received a lot of attention

[5] I will use TPC for brevity.

and approval, some of the most influential defenders of the theory are Dorothy Edgington (e.g., 1995), David Over (e.g., Over, Cruz, 2018) or Jonathan Bennett (e.g., 2003).

TPC does not aim at defining truth conditions of conditionals. It denies that conditionals have truth values at all. Instead, it defines their acceptability conditions. Acceptability (*Ac*) is understood here as aptness to be rationally accepted. So when is a conditional acceptable? According to the thesis called "Adams' thesis":[6]

AT $\text{Ac}(A \rightarrow B) = \Pr(B/A)$, provided $\Pr(A) \neq 0$

As we see, AT equals the acceptability (*Ac*) of a conditional with the conditional probability of its consequent given its antecedent ($\Pr(B/A)$). Conditional probability can be defined in different ways depending on, among other things, which probability theory we use. Adams used the standard Kolmogorov calculus and the standard definition of conditional probability (sometimes called the Ratio Formula):

RF $\Pr(B/A) = \dfrac{Pr\,(B \wedge A)}{Pr\,(A)}$

With all that in place, wherever we know the distribution of probability for the *A* and *B* we can compute how assertable is a conditional that involves these two sentences as antecedent and consequent ($A \rightarrow B$ or $B \rightarrow A$). For example, let us say that I want to know how high is the assertability of:

(9) If you jump from the fourth floor balcony, you will break your legs.

and we know the corresponding probability distribution (let us say $\Pr(9\text{a}) = 0.01$[7] and $\Pr(9\text{a} \wedge 9\text{c}) = 0.0095$) we can compute it in following way:

$$\text{Ac}(9\text{a} \rightarrow 9\text{c}) = \Pr(9\text{c}/9\text{a})$$
$$\Pr(9\text{c}/9\text{a}) = \frac{\Pr\,(9\text{a} \wedge 9\text{c})}{\Pr\,(9\text{a})}$$
$$\frac{\Pr\,(9\text{a} \wedge 9\text{c})}{\Pr\,(9\text{a})} = \frac{0.0095}{0.01} = 0.95$$

So:

[6] By → I mean the functor which connects an antecedent and a consequent in a natural language conditional.

[7] I will use *xc* and *xa* to refer to the consequent and antecedent of the example number (*x*). In the case of compound conditionals of the form, e.g., (*yaa* → *yac*) → *yc* I will use, e.g., *yaa* to refer to the antecedent of the embedded conditional. It is easy to see that for example in the case of *y*, *ya* denotes *yaa* → *yac*, I will use in such cases both labels interchangeably.

$$\mathrm{Ac}(9\mathrm{a} \rightarrow 9\mathrm{c}) = 0.95$$

This means that our conditional is highly assertable. A qualitative version of AT defines the categorical acceptability of conditionals:

(QAT) An indicative conditional "If A, B" is assertable for/acceptable to a person if and only if the person's conditional degree of belief, $P(B|A)$, is high.[8]

As we see, QAT provides a threshold of conditional probability above which the conditional is acceptable. If we accept 0.95 as high, (9) is judged by QAT to be acceptable. It is a good prediction.

2.1. Problems

TPC gives us many similarly correct results. On the other hand it has some problems, for example:

Problem 1. The denial of truth-aptness of conditionals causes many problems. For example, the reactions of participants of the experiments assessing the truth value of conditionals suggest that conditionals have truth values. When asked to assess such values (e.g., Douven, Elqayam, Singmann, van Wijnbergen-Huitink, 2020; Krzyżanowska, Collins, Hahn, 2017) they are not confused and react to conditionals as to any other truth-apt sentence. This is, unexpected if conditionals are not propositions, consider for example asking somebody about the truth-value of a clearly no truth-apt sentence, for example, a question. Such a question would be at the very least confusing. Similarly, proposals that deny the truth-aptness of conditionals have trouble with explaining embeddings of conditionals; it will be discussed in the fourth section. Other deficiencies of such an approach are discussed, for example, in (Douven, 2015; Hájek, 2012).

Problem 2. Another problem are cases of incorrect predictions. In a case where two sentences are probabilistically independent (the probability of one of the sentences does not dependent on the truth of the other) the conditional probability of the consequent given the antecedent is equal to the unconditional probability of the consequent. Therefore in such a case, if we took a conditional with very probable consequent and independent antecedent, the effect of the computation would be high acceptability. That seems unintuitive. To see this let us consider an example:

(10) If I eat an apple today, I will not inherit 1000000$ today.

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[8] This formulation is inspired by one from (Douven, Verbrugge, 2012, p. 483).

As far as I know, sadly, there is no prospect of me inheriting any dollars, so the probability of 10*c* is high, let us say 0.99. The probability of 10*a* is quite high too, let us say 0.5 (I eat an apple every other day). We can also safely assume that 10*a* does not influence 10*c* probabilistically. so $\Pr(10a \wedge 10c) = \Pr({\sim}10a \wedge 10c) = 0.495$. Now we can compute $\Pr(10c/10a)$ by RF and $\mathrm{Ac}(10a \rightarrow 10c)$ by means of AT:

$$\Pr(10c/10a) = \frac{\Pr(10a \wedge 10c)}{\Pr(10a)} = \frac{0.495}{0.5} = 0.99$$

So, (10) is highly acceptable according to TCP, which does not correspond to our intuitions. We typically do not accept such conditionals as true and if stated they would be seen as misleading.

3. A New Theory

In this section, I will propose a version of a theory that will not suffer from the problems of TPC.

Before I do that I want to note that the theory is, in a way, an idealization. The adequacy of the parameter (0.75) of the truth conditions (TC) which are the core of the theory has not been empirically tested. Therefore it is not clear how empirically adequate my proposal is. At the same time I believe that on the basis of some empirical tests, a more adequate version of my theory could be formulated. Such tests would involve subjects to make linguistic decisions involving conditionals in probabilistically transparent situations.

Another issue we should mention here is that the threshold may be sensitive to the pragmatic circumstances of the utterance. For example, a conditional whose consequent describes a dangerous event may require a lower threshold to be true. This effect can also be tested but it seems that it should be described by a pragmatic rather than a semantic theory of conditionals.

In Section 3.2 I will define the general truth conditions for conditionals and then apply them to some examples.[9]

I will assume that every sentence expresses a unique proposition,[10] which can be represented by a set of possible worlds. The probability of a proposition is the probability that it is true. By the probability of a sentence, I will understand the probability of the proposition corresponding to that sentence. It seems that ascribing probability directly to sentences would not influence the rest of my work.

[9] Their application to the cases of compound conditionals will be discussed in the fourth section.

[10] By assuming that I am simplifying by ignoring context dependence, ambiguity, etc.

3.1. Truth-Conditions

The core of the new theory is the definition of truth conditions for simple conditionals, namely:

TC The conditional $A \rightarrow B$ is true iff:

a) $\Pr(B/A) > 0.75$ and
$\Pr(B/A) > \Pr(B)$
or

b) $\Pr(A) = 0$
and false otherwise.

Both clauses of TC aim to capture different types of true conditionals.

a) captures canonical conditionals. The first conjunct aims to capture the sufficient correlation between what is described by a consequent and an antecedent of the conditional. As I already noted in face of the lack of empirical studies it is not clear how empirically adequate is the value of 0.75. The second conjunct of a) prevents TC from classifying, as true, conditionals with a non-relevantly high conditional probability (10-like). Therefore it is easy to see that (10) is not true according to TC. On the other hand, all conditionals (1) to (7) are classified as true.

The second clause is directed at Dutchman conditionals or more technically the "Ad absurdum" inferentials,[11] their classical examples are:

(11) If this is not a genuine piece of 17th century Japanese pottery, I'm a Dutchman. [(584)]
(12) If you are the new Messiah, I am Napoleon. [(587)a.]

or (8). These are not the cases of the canonic conditionals, there is no connection involved. The way we use such sentences suggests that when we use them we state that the antecedent is impossible. Consider for example:

Q1 Do you think that it is possible that it is not a genuine 17th century Japanese vase?
A1 It is impossible.

Now, if somebody uses (11) instead of A1 his answer would be equally adequate. It would preserve the original meaning. If so, we can easily incorporate

[11] Label is taken from (Declerck, Reed, 2001).

truth conditions for Dutchman conditionals by means of b).[12] Interestingly, it seems that the inferential semantics will have an analogous feature. If we assume that impossible sentences are inconsistent with the background beliefs of the speaker then, because of the explosion principle valid in classical deductive logic, we have a valid argument from an impossible antecedent to any sentence.

The consequence of adopting these truth conditions for Dutchman conditionals is that most of them are literally false. We usually use them with antecedents that are not, strictly speaking, impossible. For example, in the case of (11), it is difficult to see why "This is not a genuine piece of 17th century Japanese pottery" should actually be impossible. Still, it seems that we use expressions like A1 in similar cases where they are also literally false, so the consequence seems to be unproblematic.

How should we interpret the probability used in the definition? The definition is compatible with both subjective and objective interpretations of probability. Which of the two interpretations will be more natural depends on how realistic one is about conditionals. If somebody leans toward the suppositional view, claiming that conditionals express subjective degrees of beliefs of the speakers, then unsurprisingly the subjective interpretation of probability is natural. If somebody prefers the more objective interpretation according to which by uttering conditionals we want to claim, for example, something about regularities present in the external reality then the objective interpretation of the probability seems to be more appropriate.

There is another important issue to note here. We sometimes use conditionals with past sentences as arguments.[13] In such cases, it is natural that these sentences are actually true or false and that the probability of a false one is 0 while the probability of a true one is 1. If we combine that with TC then we will obtain a very problematic result: all conditionals with false antecedents satisfy subclause b) and are therefore true. To avoid this unwanted consequence we have to introduce a small adjustment to the theory. In the case of such conditionals, we have to use hypothetical probability instead of the actual one. We can obtain it by suspending belief in the truth of a given sentence and imagining how probable it is. In a similar way can obtain conditional probability which is needed to judge the truth value of a given conditional.

How does the theory work then? If we want to know whether the conditional is true, we compare the corresponding probability distribution with TC, first, we

[12] Another way to explain the uses of Dutchman conditionals would be to treat them as rhetorical conditionals, one which does not aim to be literally true. I have two reasons not to do it. Firstly, it seems to me that these conditionals are used in a very systematic way. That made them easy to incorporate into TC. Secondly, they naturally fill the gap in RF. It does not give us any result for conditionals with impossible antecedents.

[13] A famous example is: "If Oswald did not shoot Kennedy, someone else did" (Adams, 1970).

check if clause b) is satisfied,[14] and if it is not, we check if clause a) is. If one of the clauses is satisfied by a probability distribution, the corresponding conditional is true, and if none of them is satisfied it is false.

3.2. Examples

Let us see some applications of the proposed theory.

Example (3). Let us assume that in the case of (3), the probability that the button is pushed is 0.5.[15] If (3) is true it must be stated about the button which is responsible for turning on the alarm. The alarm has to be reliable so the probability that it goes off without pushing the button is low $\Pr(\sim 3a \wedge 3c) = (0.01)$, it is also very probable that if the button is pushed the alarm will start $\Pr(3a \wedge 3c) = (0.49)$. The probability of $3a$ is not 0 so following the procedure we have to compute the conditional probability of $3c$ given $3a$:

$$\Pr(3c/3a) = \frac{\Pr(3a \wedge 3c)}{\Pr(3a)} = \frac{0.49}{0.5} = 0.98$$

So the first part of the first clause of TC is satisfied. What about the second one? The probability of the consequent is 0.5, which is less than 0.98, so the second part of the first clause is also fulfilled, and so (3) is true.

Example (12). In all true (12)-like conditionals, the probability of the antecedent is 0. For example, in the case of $12a$, a true messiah is impossible. Thus the second clause is satisfied, therefore the conditional is true.

These results seem to be correct. Furthermore, my theory will judge (7) but not (10) as true. In the case of (10), if we plausibly prescribe probabilities, the second part of the first clause will not be satisfied (and neither will the second clause).

As we have seen, contrary to TPC, my theory accepts Dutchman conditionals but not irrelevant ones—(10)-like. If we take into consideration the way we use conditionals it seems to be an improvement.

4. Embedded Conditionals

Embedded conditionals are conditionals inside more complex sentences. They are also, arguably, the hardest cases for probabilistic theories of conditionals. In this section, I will discuss the embeddings of conditionals divided into two groups: conditionals embedded in probabilistic and non-probabilistic contexts.

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[14] If b) is satisfied we cannot compute the value of conditional probability by means of RF. That is why TPC does not give us any results in such cases.

[15] It is easy to check that this assumption does not change the result of the test, as long as the other ratios are preserved.

By the probabilistic contexts, I will understand the contexts which, when supplemented by a sentence, gain the logical value on the basis of the probability of the embedded sentence. An example could be:

(13) The probability that *x* is 0.5.

The logical value of the sentence which we obtain by the substitution will depend on the probability of the sentence which we substitute, so if we put:

(14) The outcome of the next toss with that two-euro coin will be heads.

In the place of *x* the whole sentence will be true (assuming that the coin is unbiased). The different probabilistic contexts are conditionals in light of probabilistic theories. They gain logical value on the basis of the probabilities of their arguments. I will deal with such compound conditionals in Section 4.2.

By non-probabilistic contexts I mean, quite unsurprisingly, contexts that are not probabilistic in the above sense. It is impossible to discuss conditionals in all such contexts. In the next subsection, I will focus mainly on extensional contexts.

4.1. Conditionals in Non-probabilistic Contexts

Examples of conditionals in non-probabilistic contexts are:

(15) It is true that if you press this button the fire alarm goes off. [(548)c.*]

(16) If Martha is in the kitchen, we will have dinner soon, and if Marv is in the garage, the car will be fixed tonight (Kaufmann, 2009, p. 2).

Embeddings of this type are problematic for theories that deny that conditionals have truth values (e.g., TPC). If there is no truth value for a conditional inside, e.g., conjunction, how can we determine the truth value of the whole sentence? Still, there are possible strategies for explaining such occurrences. One of them was presented in Edgington's paper (1995). She claims that for all such embeddings it is possible to express their meanings without using embedded conditionals. Sadly, the scope of this strategy is limited. It was diagnosed in (Kölbel, 2000). It seems that a similar translation is not available if a conditional is embedded within the scope of existential quantification, for example:

(17) There is a boy in my class who, if I criticize him, will get angry (Kölbel, 2000, p. 105).

In contrast to TPC, there are no problems with such embeddings in the proposed theory. According to it, conditionals are truth-apt and if we know the logical value of a given conditional we can, via truth conditions, compute the truth value of the whole sentence. For example (17) will be true iff for one of the students from the relevant class it is true that:

(18) If I criticize him, he will get angry.

As I have already noted it is impossible to discuss all non-probabilistic contexts. Still, it is easy to see that all embeddings in truth-functional contexts are easy: we just check the truth value of an embedded conditional(s), and on the basis of the truth conditions of the complex sentence determine its truth value. The cases of embeddings in extensional, but not truth-functional contexts, plausibly will also not be problematic. Strategies similar to the one used in the case of (18) are probably available there. It is hard to say anything certain about different non-probabilistic contexts like belief contexts. What is important, the main obstacle (the lack of the truth value) that makes the embeddings of conditionals difficult for TPC has been removed. Thus, as far as I know, embedded conditionals are no longer more problematic than any other embedded true-apt sentences.

4.2. Conditionals as Arguments in Probabilistic Contexts

Conditionals can appear in probabilistic contexts. The most discussed of such embeddings are compound conditionals, i.e., the conditionals with conditional antecedents or (and) consequents.

The examples of such conditionals are:

(19) If this vase will crack if it is dropped on wood, it will shatter if it is dropped on marble (Kaufmann, 2009, p. 2).

(20) If that apple is poisonous, then if you eat it you will die.

(21) If the red light is on, then if you ride another 100 kilometres your gas tank will be empty.

(22) If this house is a listed building, then if they built on a verandah, they acted illegally. [(739)a.]

If we consider the above examples it seems that we systematically use compound conditionals for example to describe dispositions. For instance, (20) describes what happens if you eat a poisonous apple. It seems that this kind of use is in line with the connection intuition. There is a connection between the antecedent which prescribes a disposition to some object and the consequent, conditional with triggering conditions in the antecedent and the effect of a disposition in the consequent. If so, probabilistic theories seem to be, in principle, able to capture cases of true compound conditionals. To do this we need to define the probability of conditionals. If we had such a definition we would be able to compute the probability of conditionals with conditional arguments just as we do it with, say, conjunction, inside of conditionals.[16]

4.2.1. Stalnaker hypothesis and its trivialization.

The first attempt to define the probability of conditional sentences aimed to do that in terms of the probability of its arguments. Unfortunately, it seems to be impossible. The most notable of such attempts is sometimes called the Stalnaker hypothesis:

SH $\Pr(A \rightarrow B) = \Pr(B/A)$

It was disproved by Lewis (1976). Moreover, Lewis by a generalization of his result shows that no similar attempt could succeed, i.e., there is no proposition x such that $P(x) = P(A \rightarrow B)$.[17]

The idea to define the probability of conditionals in terms of the probability of its arguments seems to be at least unpromising, even if we bracket the triviality proofs. In the case of all classical functors, we have such definitions. For example, we can define the probability of disjunction in terms of probability of its arguments, for example:

PD $\Pr(A \vee B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$[18]

It is easy to see that this is the case if we see it in light of the natural interpretation of the sentences about probability:

NI The probability of x is the probability that x is true.

Disjunction is a truth-functional functor. The truth of conjunction depends only on the truth of its arguments, so it is natural that the probability of the whole conjunction depends on the probability of its arguments. The story is identical for all truth-functional functors. This kind of analysis, on the other hand, is not available for the case of non-truth functional functors, e.g., we can not define $\Pr(\forall_x P(x))$ in terms of $\Pr(P(x))$.

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[16] It is easy to see that both TPC and the new version of theory can easily incorporate the conditionals of the form $(A \wedge B) \rightarrow C$ or $(A \vee B) \rightarrow C$. By means of rules like PD (see below) we can define the probability of conjunction, then we continue to compute just as in cases of conditionals with simple arguments.

[17] The argument with much weaker assumptions and the same conclusion was provided by Hájek (1994). In his later article (2012), he used structural similarities between ST and AT, as we have seen the only difference is that one thesis defines the probability of a conditional and second its acceptability, to propose a trivialization-like argument against AT and therefore TPC.

[18] Where $\Pr(A \cap B) = \Pr(B/A)\Pr(A) = \Pr(A/B)\Pr(B)$.

It seems then that if we assume that natural language conditionals are not truth-functional, then it would be at least highly surprising if an analysis such as ST would succeed. The only truth-functional candidate worth examining is material implication.[19] In face of its failure, we have to accept that the adequate truth-functional truth conditions for conditionals do not exist.

Another other definition of probability for conditionals is suggested by NI. If we combine this general rule with the truth conditions presented in Section 3.2 we will obtain the following proposal:

PC $\Pr(A \rightarrow B) = \Pr(((B/A) > 0.75) \wedge (\Pr(B/A) > \Pr(B))) \vee (\Pr(A) = 0)$

In order to use PC to compute truth values of compound conditionals, a suitable framework of second-order probability (e.g., Baron, 1987) is required. Otherwise, it is unclear how to interpret the PC and compute the probability of embedded conditionals. In light of that extending theory to be able to handle embedded conditionals goes beyond the scope of this paper.

5. Consequence of the New Theory

In this final section, I will discuss some of the consequences of my theory.

5.1. Counterexamples

There is one obvious class of simple conditionals that are used by speakers and at the same time will be systematically considered false by my theory, the so-called "biscuit conditionals", for example:

(23) If you are hungry, there is a pie in the fridge. [(628)a.]

(24) I will be in the garden if you need me. [(627)l.]

Clearly, utterances like (23) and (24) have the forms of conditionals, and at the same time, they will be judged as false by my theory (they neither involve connection which would make the first clause true nor have impossible antecedents). So it seems that they could be seen as a counterexample to my theory. On the other hand, it seems plausible that in these cases we use false conditionals to communicate some unconditional content. In the case of (23) it could be:

(23') There is a pie in the fridge. (I am telling you this in case you are hungry). [(628)b.]

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[19] It is easy to see that there are sixteen possible truth tables for a functor with two arguments. If we compare each of them with our intuitions about conditionals, the truth table of the material implication will be the most adequate.

If that is really what we communicate by means of (23), which seems plausible, then such sentences are neither true conditionals nor counterexamples to my theory.

5.2. Logical Properties of New Conditional

The detailed analysis of the logical properties of newly defined conditionals goes beyond the scope of the paper. At the same time, we can point toward some more interesting and promising properties.

First of all, the modus ponens will not be universally valid. To see that consider the following story:

> You lie in hospital because you are suffering from a serious and until now an incurable disease. Someday a nurse comes into your room and asks you if you want to take part in experimental therapy. You agreed to participate. She smiles and comments on your decision: "If you undergo the treatment, you will be just fine" (25).

If it is the case that the therapy in the story cures nine out of every ten patients who have it, does the nurse say the truth when she asserts 25? If your intuitions are like mine, you will answer in the affirmative. Moreover, if you use the new theory the result would be the same. Now we can use 25 and true sentences:

T You had the treatment.

to infer by MP that, you will be fine. But still, it can be the case that you are in the unlucky ten percent of the patients and you do not recover. This shows that there are instances of MP which do not guarantee the preservation of truth.

This may initially seem to be an unintuitive consequence but in, at least two respects, it is not so implausible. Firstly, if a conditional probability ($\Pr(xc/xa)$) of a given conditional equals 1, the inference is deductively valid. Secondly, if we adopt the probabilistic notion of validity developed in Adams' (1975) according to which an inference is valid if the probability of the conclusion is not lower than the probability of the premises, then all instances of MP are valid. This split between probabilistic and categorical validity can potentially be used to reconcile the intuitiveness of MP with proposed semantics. Working out the details of this solution goes beyond the scope of the paper.

At the same time, the theory will correctly predict valid and invalid instances of antecedent strengthening. In the case of the valid instance, for example,

(26) a) If Maureen plays the piano after 11, the neighbors complain.
b) If Maureen plays the piano after 11, and she is in her pajamas, the neighbors complain.

conditional probability of the antecedent given the consequent do not fall below the threshold necessary for conditional to be true after we add an additional conjunct to the antecedent. On the other hand in the case of the invalid instance of AS, for example:

(27) a) If Maureen plays the piano after 11, the neighbors complain.
b) If Maureen plays the piano after 11 and the neighbors are not home, the neighbors complain.

the addition lowers the conditional probability below the required level. A similar explanation is available for valid and invalid instances of transitivity.

Finally, the proposed theory does not validate Conjunctive Sufficiency principle, also called centering:
CS $A \land B \vDash A \rightarrow B$

CS is an inference that takes us from a conjunction to the conditionals from one of the conjuncts to another one. It will not be validated by proposed semantics, the second clause of the condition a) will not be satisfied in the case of many conjunctions. The fact that two sentences happened to be true at the same time does not entail that the truth of one of them promotes the truth of the second one. CS is valid in many popular semantics such as possible world semantics (e.g., Stalnaker, 1968) or three-valued semantics (e.g., Baratgin, Politzer, Over, Takahashi, 2018; Égré, Rossi, Sprenger, 2021). At the same time, some of the instances of CS seem to be counter-intuitive. For example:

(28) The clear sky is blue and Beijing is the capital of China. ⊨ If the clear sky is blue, Beijing is the capital of China.

The conjunction is true but the conditional seems to be false, there seems to be no relation between the blueness of the sky and Beijing being the capital of China. Therefore (28) seems to be a counter-example to CS. But is CS supported by the results of empirical experiments? The results are mixed but overall they seem to go against the principle. Results of the experiment presented in (Cruz, Over, Oaksford, Baratgin, 2016) support CS by showing that the way participants reacted to instances of CS is more similar to how they typically react to valid rather than invalid inferences. On the other hand, the results of Krzyżanowska, Collins, Hahn (2017), Douven, Elqayam, Singmann, van Wijnbergen-Huitink (2020), and Skovgaard-Olsen, Kellen, Hahn, Klauer (2019) go against the CS. For example, results of Krzyżanowska, Collins and Hahn (2017) suggest that the speakers expect a stronger connection between the arguments of a true conditional than between the conjuncts of true conjunction. This strongly suggests that centering is not a valid principle. The proposed semantics is not the only one that does not validate the centering, another such theory is, already mentioned, inferential semantics (e.g., Douven, Elqayam, Krzyżanowska, 2022). On the other hand, given that many of the prominent proposals validate the centering and it is not supported by the available empirical evidence it seems to be another attractive feature of the proposal.

Together with the fact that the theory incorporates the probabilistic relevance condition and explains the connection intuition, these features make the theory uniquely attractive. None of the most popular alternative theories such as possible word semantics or three-valued semantics can incorporate a similar relevance condition. Therefore, in opposition to the new theory, they are not able to explain why we do not like (10)-like conditionals.

6. Conclusion

In this concluding section, I will describe, a place of my theory in the literature devoted to indicative conditionals and gaps in my analysis which will be filed by future studies.

A similar theory was proposed by Douven (2008). The core claim of his theory, as presented in a later article (Douven, Verbrugge, 2012), is:

EST An indicative conditional "If A, B" is assertable/acceptable if and only if $\Pr(B|A)$, is not only high but also higher than $\Pr(B)$.

Both theories use the main clause which requires a conditional to have a high conditional probability of a consequent given an antecedent and the second clause which requires that an antecedent is probabilistically relevant to the consequent. Here similarities end. The main difference is that Douven's theory is pragmatic: it defines the acceptability of conditionals. Because of this difference, both theories are not competitors. If so, what is the relation between them? In the latter article, the authors do not commit themselves to any view concerning an explanation of their pragmatics: "Douven is noncommittal on whether EST is a brute fact about indicatives, or whether it follows from their truth conditions, 'or from pragmatic principles like the Gricean maxims of good conversation', or from something else altogether" (Douven, Verbrugge, 2012, p. 486).

After some minor adjustments,[20] my theory can be used to explain, in an elegant way, the proposed acceptability conditions along the line sketched in the second disjunct.

The theory is not complete. As we have seen the logical properties have to be worked out, the extension to nested conditionals is another task. I am confident that all these gaps will be filled in follow-up studies.

[20] One modification would have to involve an attitude towards Dutchman conditionals. I include them in my analysis and add to TC a sub-clause that addresses them. At the same time, Douven excludes them from his analysis.

REFERENCES

Adams, E. W. (1970). Subjunctive and Indicative Conditionals. *Foundations of Language*, *6*(1), 89–94.

Adams, E. W. (1975). *The Logic of Conditionals: An Application of Probability to Deductive Logic*. Dordrecht: D. Reidel Pub. Co.

Ajdukiewicz, K. (1956). Conditional Sentence and Material Implication. *Studia Logica*, *4*(1), 135–153.

Baratgin, J., Politzer, G., Over, D., Takahashi, T. (2018). The Psychology of Uncertainty and Three-Valued Truth Tables. *Frontiers in Psychology*, *9*(1479).

Baron, J. (1987). Second-Order Probabilities and Belief Functions. *Theory and Decision*, *23*(1), 25–36.

Bennett, J. (2003). *A Philosophical Guide to Conditionals*. Oxford: OUP.

Cruz, N., Over, D., Oaksford, M., Baratgin, J. (2016). Centering and the Meaning of Conditionals. In A. Papafragou, D. Grodner, D. Mirman, J. C. Trueswell (Eds.), *Proceedings of the 38th Annual Conference of the Cognitive Science Society* (pp. 1104–1109). Austin: Cognitive Science Society

Declerck, R., Reed, S. (2001). *Conditionals: A Comprehensive Empirical Analysis*. Berlin: Mouton de Gruyter.

Douven, I. (2008). The Evidential Support Theory of Conditionals. *Synthese*, *164*(1), 19–44.

Douven, I. (2015). *The Epistemology of Indicative Conditionals: Formal and Empirical Approaches*. Cambridge: Cambridge University Press.

Douven, I., Elqayam, S., Krzyżanowska, K. (2022). Inferentialism: A Manifesto. In *Conditionals: Logic, Linguistics, and Psychology*. Retrieved from: https://hal.archives-ouvertes.fr/hal-03508608/document

Douven, I., Elqayam, S., Singmann, H., van Wijnbergen-Huitink, J. (2020). Conditionals and Inferential Connections: Toward a New Semantics. *Thinking and Reasoning*, *26*(3), 311–351.

Douven, I., Verbrugge, S. (2012). Indicatives, Concessives, and Evidential Support. *Thinking and Reasoning*, *18*(4), 480–499.

Edgington, D. (1995). On Conditionals. *Mind*, *104*(414), 235–329.

Égré, P., Rossi, L., Sprenger, J. (2021). De Finettian Logics of Indicative Conditionals Part I: Trivalent Semantics and Validity. *Journal of Philosophical Logic*, *50*.

Grice, H. P. (1989). *Studies in the Way of Words*. Cambridge, MA: HUP.

Hájek, A. (1994). Triviality on the Cheap? In E. Eells, B. Skyrms, E. W. Adams (Eds.), *Probability and Conditionals: Belief Revision and Rational Decision* (pp. 113–40). Cambridge: Cambridge University Press.

Hájek, A. (2012). The Fall of "Adams' Thesis"? *Journal of Logic, Language and Information*, *21*(2), 145–161.

Kaufmann, S. (2009). Conditionals Right and Left: Probabilities for the Whole Family. *Journal of Philosophical Logic*, *38*(1), 1–53.

Kölbel, M. (2000). Edgington on Compounds of Conditionals. *Mind*, *109*(433), 97–108.

Krzyżanowska, K., Wenmackers, S., Douven, I. (2014). Rethinking Gibbard's Riverboat Argument. *Studia Logica*, *102*(4), 771–792.

Krzyżanowska, K., Collins, P. J., Hahn, U. (2017). Between A Conditional Antecedent and Its Consequent: Discourse Coherence vs. Probabilistic Relevance. *Cognition*, *164*, 199–205.

Lewis, D. (1976). Probabilities of Conditionals and Conditional Probabilities. *Philosophical Review*, *85*(3), 297–315.

Over, D., Cruz, N. (2018). Probabilistic Accounts of Conditional Reasoning. In L. J. Ball, V. A. Thompson (Eds.), *The Routledge International Handbook of Thinking and Reasoning* (pp. 434–450). Milton Park: Routledge/Taylor & Francis Group.

Skovgaard-Olsen, N., Kellen, D., Hahn, U., Klauer, K. (2019). Norm Conflicts and Conditionals. *Psychological Review*, *126*(5), 611–633

Stalnaker, R. (1968). A Theory of Conditionals. In N. Rescher (Ed.), *Studies in Logical Theory* (American Philosophical Quarterly Monographs 2, pp. 98–112). Oxford: Blackwell.

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Article

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CONTEXT-INDEXED COUNTERFACTUALS

SUMMARY: It is commonly believed that the role of context cannot be ignored in the analysis of conditionals, and counterfactuals in particular. On truth conditional accounts involving possible worlds semantics, conditionals have been analysed as expressions of relative necessity: "If *A*, then *B*" is true at some world *w* if *B* is true at all the *A*-worlds deemed relevant to the evaluation of the conditional at *w*. A drawback of this approach is that for the evaluation of conditionals with the same antecedents at some world, the same worlds are deemed as relevant for all occasions of utterance. But surely this is inadequate, if shifts of contexts between occasions are to be accounted for. Both the linguistic and logical implications of this defect are discussed, and in order to overcome it a modification of David Lewis' ordering semantics for counterfactuals is developed for a modified language. I follow Lewis by letting contexts determine comparative similarity assignments, and show that the addition of syntactic context parameters (context indices) to the language gives the freedom required to switch between sets of relevant antecedent worlds from occasion to occasion by choosing the corresponding similarity assignment accordingly. Thus an account that extends Lewis' analysis of a language containing a single counterfactual connective > to a language containing infinitely many counterfactual connectives $>_c$, each indexed by a different context name *c*, overcomes the limitations of traditional analyses. Finally it is also shown that these traditional accounts can be recovered from the modified account if certain contextual restrictions are in place.

KEYWORDS: ordering semantics, counterfactuals, comparative similarity, context, contextual information.

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Introduction

On many possible world semantics for conditional logics, which famously include Stalnaker-Lewis truth conditional accounts, only the world of evaluation and the antecedent are considered in selecting worlds that are deemed relevant to determining the truth value of a conditional.[1] But that results in the underlying context being fixed for all occasions—even when contextual considerations underlying the evaluation of the uttered counterfactuals on various occasions may vary.[2] Alternative approaches go some of the way toward resolving this inadequacy by appealing to a difference in the consequents associated with counterfactuals with the same antecedent, but nevertheless such approaches are still limited to evaluating any conditional with a fixed truth value on any occasion. In this article I propose an analysis of a language that makes appropriate explicit access to the intended context available by introducing explicit names for contexts that index the counterfactual connective. That is, I give an account of a contextualized counterfactual of the form “In context *C*: If it were the case that…, then it would be the case that…”. Although the proposal is largely based on David Lewis’ analyses of counterfactuals, it does not require that any particular logic of conditionals should serve as its basis—rather, it is intended as a general prescription for contextualizing a conditional language. The contextualization can be applied to the weakest of conditional logics. That is, the method in the manner described is generalizable (extendable) to the weakest of conditional logics, e.g., the system **CE** (Chellas, 1975, p. 138; Nute, 1980, p. 53; Weiss, 2018, p. 15). The advantage of working with stronger logics and ordering semantics stems from existing results, due to Lewis (1981), concerning the properties of ordering frames that facilitate fashioning and implementing a notion of contextual information preservation, which is central to the semantics of the proposed account.

There are three key results concerning the account proposed in this article, which can only be described informally at this point. The first result is at the level of Lewis’ ordering semantics for counterfactuals, and it concerns semantic (truth preserving) properties of a certain class of ordering frames (ordering frame

[1] The tradition of analysis contested in this article refers mainly to Stalnaker, Lewis, and Gabbay all of whom offer truth conditional accounts of conditionals involving possible worlds semantics (Nute, 1981, Section 4). More generally, this concerns analyses that take only the semantic content provided by the world of evaluation and the antecedent (and the consequent, in Gabbay’s case) in order to evaluate the conditional. The most general of those include—using Nute’s (1981, Chapter 3) classification terminology—conditional logics characterized by world selection function models (WS-models), systems of spheres models (SOS-models), relational models (R-models), class selection function models (CS-models), and neighbourhood models (N-models). The most well-known of those include analyses given in (Chellas, 1975; Lewis, 1973; Montague, 1970; Scott, 1970; Stalnaker, 1968).

[2] To clarify the terminology, an *occasion* of utterance (consideration, or evaluation) of an expression is the *time and place* of such an utterance (consideration, or evaluation). What should be clear is that in any given possible world there are numerous occasions.

refinements), and its importance stems from the role it plays in establishing key results of the modified (contextualized) account. The two subsequent results concern the modified account, which is developed as an analysis of a language containing context-indexed conditionals (contextualized language). Informally, the first states that if discourse is restricted to a single context, then the model theory of the modified account reduces (as it would be expected) exactly to the Stalnaker-Lewis' analysis of counterfactuals, in particular, extensions of **VC**. In this sense, the modified account is really just an extension of Stalnaker-Lewis type of analyses—it is *equivalent* to those accounts when dealing with sets of formulae that contain counterfactual connectives ranging over a single context index, but it extends those accounts by offering a model theory that can handle evaluating, and making inferences over sets of formulae containing counterfactual connectives whose context indices vary. The second, and more general result concerns the recovery of Stalnaker-Lewis analysis on the modified account, if certain contextual information preservation conditions are satisfied. Namely, part of the logic given by the **VC** semantic consequence relation can be preserved on the proposed account for those inferences (ranging over the contextualized language) where the context index of the conclusion is said to preserve some of the mutual contextual information of the context indices over which the premises range. The second result is applied in fashioning a logic of contextualized counterfactuals, offered in the form of a semantic consequence. It is intended as a logic that is sensitive to explicit contextual content. Contextual validity is strengthened by adding the requirement of contextual information preservation to the standard requirement of truth preservation at all possible worlds.

1. Counterfactuals and Context

Counterfactuals are expressions of the form "If it were the case that A, then it would be the case that B" (formally, $A > B$), where A and B are propositions. It is commonly believed that they are notoriously context sensitive. Take a well-known example:

1. If Caesar had been in command, he would have used the atom bomb.
2. If Caesar had been in command, he would have used catapults.[3]

Intuitively, the truth of each depends on contextual background assumptions. Clearly, for the first statement to be true, we require contexts where Caesar's knowledge of modern warfare is assumed to be in line with the military knowledge of a modern military general, whereas for the second to be true, no such contextual background assumption is required.[4] David Lewis (1973, pp. 66–67) approaches this contention by proposing a *rule of accommodation*,

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[3] Quine (1960, p. 22) bases this example on similar ones given by Goodman (1954).

[4] Gabbay (1972, pp. 98–99) argues essentially along the same lines.

whereby the uttered counterfactual is taken as being asserted, and then context is called upon in resolving the vagueness of the comparative similarity in favor of the truth of the uttered counterfactual (for the purposes of the present discussion I will say that *a context justifies the assertion of a given conditional* to refer to the aforementioned role of context when employing the rule of accommodation).[5] However, the key drawback of this solution is that for any world of evaluation and explicit antecedent, a single context is called upon to justify the assertion of a counterfactual on any occasion. That is, a single context is fixed for all occasions. The formal semantics of Lewis' account and other, aforementioned analyses is clear in that regard. Donald Nute elucidates this fact as follows:

> SOS-models involve functions which take only possible worlds as arguments, while both SC-models and WS-models involve functions which take both possible worlds and sentences-qua-antecedents as arguments. [However] the evaluation of two conditionals with the same antecedent may require consideration of different sets of situations. Any semantics which takes into account only the antecedent of the conditional and the situation of the speaker in determining the situations to be considered in hypothetical deliberation does not explicitly recognize this fact. (1981, p. 73)[6]

Another way of seeing this major drawback is by highlighting a fundamental feature, pointed out by Chellas (1975, p. 138), that those possible world conditional analyses have in common—namely, of the conditionals being conceived of as expressions of relative necessity (for a detailed overview of such analyses, see Chellas, 1975; Priest, 2008, Sections 5.3 and 5.5; Weiss, 2018). This has the following consequence—when evaluating the truth of a counterfactual at some possible world *w*, the antecedent effectively acts as *restricted necessity operator*, making accessible only those possible worlds that have the features we take to be relevant to our deliberations in evaluating the conditional. But because only *w* and the explicit antecedent are employed in the determination of that restriction on those accounts, it is fixed for all occasions for conditionals with the same antecedent. But surely the features we take to be relevant to our deliberations in evaluating the conditional are not the same for all occasions, since contextual considerations underlying each occasion are bound to change.

In what follows I will argue why the aforementioned approaches (which in the current discussion shall be referred to as the *class of contested accounts*) are inadequate if we take the role of context seriously. The objection can be viewed as having two components. The first part is mainly linguistic, focusing on the inadequacy of analyzing sets of asserted conditionals accross various occasions

[5] Lewis (1986, p. 251) maintains his approach and expresses this idea succinctly: "[t]here is a rule of accommodation: what you say makes itself true, if at all possible, by creating a context that selects the relevant features so as to make it true".

[6] Nute (1981, pp. 72, 76). Footnote 1 of the current paper disambiguates the acronyms used by Nute in the cited fragment.

in a manner that accounts for contextual differences underlying those occasions and consequently distinct justifications of the assertions. The second part of the objection focuses on the implications that such inadequacy has for the logic of counterfactuals, i.e., making inferences from sets of statements that include counterfactuals.

Accounts from the aforementioned, contested class fare fine when dealing with conditionals considered in isolation, however difficulties appear when we consider sets of conditionals, and in particular, inferences containing conditionals. It is clear from Lewis' formal semantics (1973; 1981) that asserting a set of conditionals across more than one occasion at any possible world is restricted to a single assertion-justifying context (modelled by a *single* similarity assignment to that world). However, given two conditionals with explicitly identical antecedents, we may wish to call upon different contexts (not just a single one) on distinct occasions to justify our assertions of either conditionals. For example, we may wish to have our assertion of (1) justified on one occasion by a context that does not justify the assertion of (2), and on another occasion have the assertion of (2) justified by a context that does not justify the assertion of (1).[7] To put it another way, on any two occasions we may wish to be free to assert conditionals with the same antecedent for different reasons (by recourse to different contexts that accordingly justify each assertion) or we may even wish to assert the same conditional for different reasons on two occasions, and as such not be restricted to relying on a single context in providing the corresponding justifications for those assertions.

Presently I shall give examples that aim to illustrate the inadequacy of the aforementioned accounts when tasked with a treatment that is supposed to account for context sensitivity when dealing with *sets* of counterfactuals. Let us first consider the following pair of counteridenticals given by Goodman (1954). Here the antecedents are the same, but their consequents are contradictory, on the assumed identity.

3. If I was Julius Caesar, I would not be alive in the 21st century.
4. If I was Julius Caesar, he would be alive in the 21st century (Goodman, 1983, p. 6).

Imagine asserting (3) on one occasion and asserting (4) on another occasion, at the same possible world. Those assertions are justified by recourse to different contexts on those occasions—clearly for the truth of (3) I assume being alive in the 1st century BCE, whereas no such assumption is required for the truth of (4)—but on the traditional accounts only a single context is available for both of

[7] Berto (2017, Section 5) essentially agrees with this view, the interpretation difference being that when Lewis speaks of *assertions* Berto speaks of *acts of imagination*, and when Lewis speaks of *explicit antecedents*, Berto speaks of *explicit content of the imagination acts*.

those occasions, i.e., the same set of antecedent worlds is considered as relevant in the evaluation of both conditionals. This is clearly inadequate. It seems that both (3) and (4) can be asserted or at least they can both be heard as true, albeit *according to different contexts* (see also Priest, 2016, p. 4). However, on accounts in the contested class of analyses, such pairs cannot be both evaluated as true at the same world, since their formal model theories allow only a single context to underlie the evaluation of any counterfactual with the same antecedent on any occasion, which means that on possible world analyses at most one conditional in the above pair can be evaluated as true at the same world.[8] Another interesting class of examples similar to (3) and (4) comes from a widespread phenomenon of contentious pairs of indicative conditionals known in the literature as "Gibbardian Stand-Offs", whereby its seems clear that there are good reasons for the truth (or assertion) of two conditionals with identical antecedents yet contradictory consequents, albeit each in its own context. I argue in Section 3.2 that my proposal can also be applied in offering a solution to these phenomena burdening the indicative conditional.

Those limitations have direct implications for the logic of counterfactuals, as becomes evident from the inference forms that the presence of those limitations is responsible for validating. Let us consider the example given by Quine again:

1. If Caesar had been in command, he would have used the atom bomb.
2. If Caesar had been in command, he would have used catapults.

As it has been already said, the kind of relevant assumptions required for the truth of (1) are not the same as those required for the truth of (2). There may be good reasons to assert (1) in some contexts and (2) in others. Moreover we may wish to assert (or evaluate as true) both on a single occasion, yet with recourse to distinct contexts that justify the assertion of each. However, the truth of both (1) and (2) should not entail the truth of:

5. If Caesar had been in command, he would have used catapults and the atom bomb.

Sure, there may exist a strange context that accounts for such idiosyncratic decisions (after all, it is possible to use both nukes and catapults), but inferring (5) from (1) and (2) should not be an automatic entailment, because clearly that depends on what contexts have been employed in the justification of (1) and (2), i.e., presumably, not always a single strange context (see also Berto, 2014, p. 113). However, all accounts in the contested class, that evaluate both (1) and (2) as true at some world are committed to evaluating (5) as true at that world.

[8] On analyses that invalidate conditional excluded middle, i.e., $(A > B) \vee (A > \sim B)$, there may be the third possibility of both being evaluated as false, e.g., this is one of the differences between Stalnaker's and Lewis' accounts.

This stems from the fact that whenever sets of conditionals with the same antecedent are modelled as jointly true at some world *w*, the set of relevant antecedent worlds employed in the evaluation of those conditionals at *w* is the same for all those conditionals. This becomes even more starkly evident in the example from Goodman. As discussed earlier, we may wish to accept both (3) and (4) as true, on different occasions, but we would never accept the truth of:

> If I was Julius Caesar, I (Julius Caesar) would and would not be alive in the 21st century.

It is not surprising that such analyses validate the inference of $A > (B \wedge C)$ from both $A > B$ and $A > C$ (henceforth referred to as *Adjunction of Consequents*) or equivalently have $(A > B \wedge A > C) \supset A > (B \wedge C)$ as the corresponding axiom in their respective proof theories.[9] Note that since on the contested accounts (3) and (4) can never be both evaluated as true, the inference goes through vacuously.

Gabbay's (1972) analysis of conditionals has one apparent advantage over the analyses in the contested class as it offers a semantic counterpart for the fact, which we have observed, that the evaluation of two conditionals with the same antecedent may require consideration of different sets of situations (Nute, 1980, p. 75). That semantic counterpart is the consequent, which is employed as an additional parameter that allows accounting for a potential context shift on any single occasion of utterance by considering different sets of worlds in the evaluation of conditionals with the same antecedents (for a more formal explanation, see Popieluch, 2019, pp. 32–36). However, as Nute (1980, p. 76) observes, Gabbay's analysis much like the analyses from the contested class will give a single, determinate truth value to the conditional, regardless of the contextual circumstances under which the conditional is evaluated, i.e., the same truth value for all occasions. Nute observes that there may be a relevant difference in the occasions of evaluation, even when both the antecedent and the consequent of the conditional remain the same, however Gabbay's formal semantics fails to offer a semantic mechanism that would allow flexibility in evaluating a conditional in a manner that accounts for distinct contextual considerations more than occasion. So in this sense Gabbay's account fares no better than Lewis'.

In the next section an analysis of counterfactuals is presented that avoids both the linguistic and logical issues described above. The presentation of the aforementioned account intends to be neutral with regard to the matter of subjunctive-indicative distinction and the discussion accompanying the presentation makes no commitments with regard to whether that distinction is fundamental or only apparent, and consequently whether there is a single, unifying analysis for both, or not. Rather, the aim of the article is to offer an analysis that can be applicable whenever context related issues do arise, or have been argued to arise. Because

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[9] Such inferences are valid on Lewis's logic VC and its extensions, which include Stalnaker's logic of conditionals.

subjunctives have been mostly burdened with such issues, much of the article does focus on counterfactuals. Moreover, because the modified account builds on Lewis' account, which is uniquely tailored to counterfactual conditionals (generally uttered in the subjunctive mood), the focus of the proposal has been mostly confined to counterfactuals, but also in the last section an important application of the proposed account to indicative conditionals is discussed. It concerns phenomena of "Gibbardian Stand-Offs", which have been identified by a number of authors to be essentially context related in nature (e.g., Bennett, 2003; Priest, 2016; Santos, 2018).

2. An Alternative

The alternative account, proposed in this article, is developed as a modification of ordering semantics for counterfactuals that proceeds by (i) expanding the formal language by substituting the single conditional connective > with an entire family of indexed connectives $\{>_c : c \in \mathcal{C}\}$, ranging over an index set $\mathcal{C}$ and (ii) subsequently a modified model theory is provided for the evaluation of the logical value of expressions $A >_c B$ interpreted as "In context c: If it were the case that A, then then it would be the case that B". Since the modified account is offered as a modification of ordering semantics for counterfactuals given by Lewis (1974; 1981), I begin by laying out the formal details of the latter. This is required since it is within that formalism that key concepts, such as *ordering frame refinements* are defined, which underlie the main results and the formal foundation for the semantic consequence of the modified account.

The culmination of the modified account is a logic of contextualized counterfactuals, offered in the form of a semantic consequence relation. The idea of *contextual validity*, adds to the standard requirement of *truth preservation* at all possible worlds a second requirement of *contextual information preservation*. A very similar idea—in terms of preserving imported information throughout an inference—is explored in Priest (2016, p. 8). Yet another approach, which proceeds by contextually restricting inferences via a language that contains a certain family of context indexed intensional connectives is outlined in Berto (2017, p. 11).[10]

2.1. Ordering Semantics for Counterfactuals

The resulting logic **CS** that is endorsed in this section is much like Lewis' preferred account save for strict centering being replaced with a weaker centering condition. That is, **CS** is just the logic that Lewis (1973) calls **VW**, which is obtained from his preferred system **VC** (commonly referred to as **C1**) by replacing the *strict centering* condition with the *weak centering* condition, or equivalently, removing the axiom $(A \wedge B) \supset (A > B)$ from the deductive system for **VC**.

[10] For a discussion outlining the similarity between Berto's context indexation suggestion and the approach offered in this paper, see (Popieluch, 2019, pp. 38–40).

2.1.1. Formal language.

Let us start with the basic ingredients for our language, i.e., a set of propositional variables $PV = \{p_n : n \in \mathbb{N}\}$ the elements of which shall be denoted with lowercase Roman letters (p, q, r, …) or subscripted lowercase Roman p's ($p_1, p_2, \ldots, p_k, \ldots$), or lowercase Greek letters (φ, ψ, χ, …); unary connectives: ~ (negation), □ (necessity), ◊ (possibility); and binary connectives: ∧ (conjunction), ∨ (disjunction), ⊃ (material conditional), > (counterfactual conditional). For the metalanguage, upper case letters (A, B, C, …) shall be used as variables ranging over complex formulae and propositional variables.

Definition 1.1. Define the language of interest, denoted $\mathcal{L}$, to be the set: {~, □, ◊, ∧, ∨, ⊃, >}.

Now we define the set of well-formed formulae.[11]

Definition 1.2. Let *For* be the smallest set closed under the following well-formed formula formation rules:

B: All propositional variables are wffs, i.e., $PV \subseteq For$.

R1: If $A \in For$ then $\{{\sim}A, \Box A, \Diamond A\} \subseteq For$.

R2: If $\{A, B\} \subseteq For$ then $\{A \wedge B, A \vee B, A \supset B, A > B\} \subseteq For$.

Definition 1.3. It will be helpful to define the subset of *For* that contains all and only formulae that contain occurrences of >. Denote that subset with $For_{>}$.

Definition 1.4. Denote the set $For \backslash For_{>}$ with For_0, which is just the set of wffs of the basic modal language.

2.1.2. Comparative similarity.

In order to establish the relations in our semantics, we need to introduce their intended meaning and basic properties. The systems of spheres are just a convenient, and intuitive way for representing information about the comparative similarity of worlds (Lewis, 1973, p. 48). We can do the same, directly in terms of comparative similarity of worlds, together with accessibility. To make this explicit let us consider the following definitions.

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[11] E.g., the counterfactual "If kangaroos had no tails, they would topple over" would have the form: $p > q$, where p stands for "kangaroos have no tails" and q stands for "kangaroos topple over".

Definition 2.1. A binary relation $R \subseteq S \times S$ on a set S, denoted by $\lesssim$, is a *preorder* iff it is:

(1) transitive: $\forall x,y,z \in S\, ((x \lesssim y \wedge y \lesssim z) \longrightarrow x \lesssim z)$.

(2) reflexive: $\forall x \in S\, (x \lesssim x)$.

If $\lesssim$ satisfies (1), (2), and (3), it is a *total preorder* (also called a *non-strict weak order*).

(3) totality: $\forall x,y \in S\, (x \lesssim y \vee y \lesssim x)$. [12]

Definition 2.2. For any preorder $\lesssim$, denote $(x, y) \notin \lesssim$, i.e., "it is not the case that $x \lesssim y$" with $y < x$, and let us write $x \sim y$ to mean that both $x \lesssim y$ and $y \lesssim x$.

Lemma 2.1. If $\lesssim$ is a *preorder* on S then for no $x \in S$: $x < x$.

Proof. This follows directly from reflexivity of $\lesssim$, i.e., $x < x$ means $(x, x) \notin \lesssim$, contradicting reflexivity of $\lesssim$. □

Lemma 2.2. If $\lesssim$ is a *total preorder* on S, then for all $x, y \in S$:

(i) $x < y$ iff $(x, y) \in \lesssim$ and $(y, x) \notin \lesssim$,

(ii) $x \lesssim y$ iff $x < y$ or $x \sim y$.

Proof. (i) $(y, x) \notin \lesssim$ follows from definition of $x < y$, and $(x, y) \in \lesssim$ follows from totality of $\lesssim$. (ii) Given totality, either $(x, y) \in \lesssim$ and $(y, x) \notin \lesssim$ or both $(x, y) \in \lesssim$ and $(y, x) \in \lesssim$. The third, totality satisfying option $(x, y) \notin \lesssim$ and $(y, x) \in \lesssim$ is clearly impossible. □

My definition of ordering frames based on comparative similarity closely follows the definition of a *comparative similarity system* in Lewis (1973, p. 48), save for the condition corresponding to what Lewis calls *centering*, i.e.,

(CS3.1) The element i is $<_i$-minimal: $\forall j \in W (j \neq i \longrightarrow i <_i j)$,

which I replace with a weaker condition (CS3) corresponding to *weak centering*.

Definition 2.3. An *ordering frame* based on comparative similarity is a pair $(W, \lesssim)$, where W is a nonempty set and $\lesssim\colon W \longrightarrow \wp(W) \times \wp(W \times W)$ is a function that assigns to each $i \in W$ a pair $(S_i, \lesssim_i)$, consisting of a set $S_i \subseteq W$, regarded as the set of worlds accessible from i, and a binary relation $\lesssim_i$ on W, regarded as the

[12] Lewis (1973, p. 48) refers to this property as "strongly connected".

ordering of worlds in respect of their comparative similarity to i and satisfying the following conditions, for each $i \in W$:

(CS1) $\lesssim_i$ is a total preorder on S_i.

(CS2) i is self-accessible: $i \in S_i$.

(CS3) i is $\lesssim_i$-minimal: $\forall j \in W(i \lesssim_i j)$.

(CS4) Inaccessible worlds are $\lesssim_i$-maximal: $\forall j,k \in W(k \notin S_i \longrightarrow j \lesssim_i k)$.

(CS5) Accessible worlds are more similar to i than inaccessible worlds:

$$\forall j,k \in W((j \in S_i \wedge k \notin S_i) \longrightarrow j <_i k)$$

On the intended interpretation, elements of W are possible worlds, S_i is regarded as the set of worlds accessible from i, and $\lesssim_i$ is regarded as the ordering of worlds in respect of their comparative similarity to i, with the following intended meaning:

$j \lesssim_i k$: j is at least as similar to i as k is,

$j <_i k$: j is more similar to i than k is,

$j \sim_i k$: j and k are equally similar to i.[13]

Definition 2.4. Denote the *class of ordering frames from Definition 2.3* by **CS**.

Note that since centering implies weak centering, the class of ordering frames satisfying (CS3.1) instead of (CS3) is a proper subclass of **CS**.[14]

Definition 2.5. Given some $F \in$ **CS**, let W^F denote the domain of F and let $\lesssim^F$ denote F's ordering assignment on F's domain, i.e., $W^F \longrightarrow \wp(W^F) \times \wp(W^F \times W^F)$ as defined in 2.3. Also, let S_i^F and $\lesssim_i^F$ denote the elements of the image $(S_i^F, \lesssim_i^F)$ of $i \in W^F$ under $\lesssim^F$.

Definition 2.6. A *model based on comparative similarity* is the triple $(W, \lesssim, V)$ such that $(W, \lesssim)$ is an ordering frame and for each $i \in W$, V_i: $PV \longrightarrow \{0, 1\}$ is a function from PV to $\{0, 1\}$. Informally we think of $\{i \in W: V_i(p) = 1\}$ as the set of worlds in the model where is p true, and $\{i \in W: V_i(p) = 0\}$ as the set of worlds in the model where is p false.

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[13] Lewis' (1981, p. 220) definition of $\sim_i$ in terms of a strict comparative similarity relation $<_i$ is logically equivalent to the one he gave earlier, in (Lewis, 1973, p. 48)—the one I choose to use in this article. In terms of $<_i$ the comparative similarity equivalence $\sim_i$ is defined as follows: $j \sim_i k$: neither $j <_i k$ nor $j <_i k$.

[14] Since, if $j <_i^F k$, then $j \lesssim_i^F k$ for any $i, j, k \in W$, by totality and definition of $<_i^F$.

Truth in a model is defined in terms the satisfiability relation $\Vdash\ \subseteq W \times For$. We read $i \Vdash A$ as "A is true at i". Given a model $(W, \lesssim, V)$ and any $i \in W$, define $\Vdash$ as follows:

(1)	$i \Vdash p$	iff	$V_i(p) = 1$
(2)	$i \Vdash {\sim}A$	iff	not $i \Vdash A$
(3)	$i \Vdash A \wedge B$	iff	$i \Vdash A$ and $i \Vdash B$
(4)	$i \Vdash A \vee B$	iff	$i \Vdash A$ or $i \Vdash B$
(5)	$i \Vdash A \supset B$	iff	$i \Vdash {\sim}A$ or $i \Vdash B$
(6)	$i \Vdash \Box A$	iff	$\forall j \in W: j \Vdash A$
(7)	$i \Vdash \Diamond A$	iff	$\exists j \in W: j \Vdash A$
(8)	$i \Vdash A > B$	iff	(i) ${\sim}\exists k \in S_i: k \Vdash A$, or
			(ii) $\exists k \in S_i: k \Vdash A$ and $\forall j \in S_i\ (j \lesssim_i k \longrightarrow j \Vdash A \supset B)$

For convenience, let us introduce the following notation: $i \Vdash \Sigma$ iff $i \Vdash A$ for all $A \in \Sigma$.

When we want to explicitly refer to truth at a world in some model $\mathfrak{A}$, we shall employ the following notation: $\mathfrak{A}, i \Vdash A$ and $\mathfrak{A}, i \Vdash \Sigma$. Also, write $\mathfrak{A} \Vdash A$ when $\mathfrak{A}, i \Vdash A$ for all $i \in W^{\mathfrak{A}}$.

Definition 2.7. It will also be convenient to define $[A]^{\mathfrak{A}} := \{i \in W: \mathfrak{A}, i \Vdash A\}$ for any model $\mathfrak{A}$ with domain W. The superscript will be omitted in cases when its absence will not lead to ambiguity.

Definition 2.8. Let $\vDash_{CS}\ \subseteq \wp(For) \times For$, and define $\Sigma \vDash_{CS} A$ iff for all models $(W, \lesssim, V)$, and all $i \in W$, if $i \Vdash B$ for all $B \in \Sigma$, then $i \Vdash A$. We say an inference from Σ to A is valid iff $\Sigma \vDash_{CS} A$. That is, valid inference is defined as truth preservation at all worlds in all **CS**-models. A formula $A \in For$ is said to be valid iff $\emptyset \vDash_{CS} A$. Call this logic **CS**.

Note that since the truth conditions for $\Box$ and $\Diamond$ formulae are defined in terms of unrestricted quantification over possible worlds, i.e., only >-formulae truth conditions contain accessibility restrictions, the above validity conditions give the modal logic **S5** for the basic modal language.

Just as we have relativized formula validity to a model $\mathfrak{A} \Vdash A$ it will be of use to define valid inference relativized to a model.

Definition 2.9. Let $\vDash_{\mathfrak{A}}\ \subseteq \wp(For) \times For$, and given a **CS** model $\mathfrak{A} = (W, \lesssim, V)$, write:

(i) $\vDash_{\mathfrak{A}} A$ iff $\mathfrak{A} \Vdash A$
(ii) $\Sigma \vDash_{\mathfrak{A}} A$ iff for all $i \in W$, if $\mathfrak{A}, i \Vdash B$ for all $B \in \Sigma$, then $\mathfrak{A}, i \Vdash A$.

This allows us to give a more succinct definition of semantic consequence:

$$\Sigma \vDash_{\mathbf{CS}} A \text{ iff for all } \mathbf{CS} \text{ models } \mathfrak{A}\text{: } \Sigma \vDash_{\mathfrak{A}} A$$

Note that it is immediate from the above definitions that $\vDash_{\mathbf{CS}} \subseteq \vDash_{\mathfrak{A}}$, for any **CS** model $\mathfrak{A}$.

2.1.3. Ordering frame refinements and dilutions.

Let us now turn to defining ordering frame refinements and dilutions, which are the key protagonists in the account of ordering semantics presented here.[15]

Definition 3.1. Let $\mathcal{R} \subseteq \mathbf{CS} \times \mathbf{CS}$ and call an ordering frame G a *refinement* of ordering frame F iff $(F, G) \in \mathcal{R}$. And define $(F, G) \in \mathcal{R}$ iff:

(i) $W^G = W^F$,
and for all $i \in W^F$:
(ii) $\lesssim_i^G \subseteq \lesssim_i^F$
(iii) $S_i^G = S_i^F$

Definition 3.1.1. A *proper refinement* of F is a refinement G, such that $G \neq F$.

Definition 3.1.2. Let $\mathcal{R}[F] \coloneqq \{G \in \mathbf{CS}\colon (F, G) \in \mathcal{R}\}$ denote the *image* of F under $\mathcal{R}$, i.e., the set of all refinements of F.

Definition 3.2. Let $\mathcal{D} \subseteq \mathbf{CS} \times \mathbf{CS}$ and call an ordering frame G a *dilution* of ordering frame F iff $(F, G) \in \mathcal{D}$. And define $(F, G) \in \mathcal{D}$ iff:

(i) $W^G = W^F$,
and for all $i \in W^F$:
(ii) $\lesssim_i^F \subseteq \lesssim_i^G$
(iii) $S_i^G = S_i^F$

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[15] The essential idea of refinements is based on (Lewis, 1981, pp. 226–227). However, Lewis (1981) defines refinements on strict preorder relations: if $j <_i^F k$, then $j <_i^G k$ (where G is a refinement of F). Given the way I have defined refinements (using total preorders) Lewis' definition is a derived property of refinements, i.e., Lemma 4.1.

Definition 3.2.1. A *proper dilution* of F is a dilution G of F, such that $G \neq F$.

Note: the orderings of refinements and dilutions are *total*, by definition of ordering frames.

Definition 3.2.2. Let $\mathcal{D}[F] \coloneqq \{G \in \mathbf{CS}\colon (F, G) \in \mathcal{D}\}$ denote the *image* of F under $\mathcal{D}$, i.e., the set of all dilutions of F.

2.1.4. Elementary properties of refinements and dilutions.

Now we prove some elementary yet crucial properties of refinements and dilutions. Frame refinements *preserve the strict ordering* of original ordering frames in the following sense:

Lemma 4.1. If G is a refinement of F, then if $j <_i^F k$ for any i, j, k according to some comparative similarity assignment $(S_i^F, \lesssim_i^F)$, then $j <_i^G k$ according to $(S_i^G, \lesssim_i^G)$.

Proof. It suffices to note that, since $\lesssim_i^F$ is total and $\lesssim_i^G \subseteq \lesssim_i^F$ for each i, then if $(j, k) \in \lesssim_i^F$ and $(k, j) \notin \lesssim_i^F$, i.e., $j <_i^F k$, then it follows that both $(j, k) \in \lesssim_i^G$ and $(k, j) \notin \lesssim_i^G$, i.e., $j <_i^G k$. Denying $(k, j) \notin \lesssim_i^G$ contradicts the subset property, and denying $(j, k) \in \lesssim_i^G$ contradicts totality. □

We have a dual result to Lemma 4.1 for frame dilutions. That is, frame dilutions *preserve the non-strict ordering* of original ordering frames in the following sense:

Lemma 4.2. If G is a dilution of F then if $j \lesssim_i^F k$ for any i, j, k according to some comparative similarity assignment $(S_i^F, \lesssim_i^F)$, then $j \lesssim_i^F k$ according to $(S_i^G, \lesssim_i^G)$.

Proof. It suffices to observe that, since $\lesssim_i^F \subseteq \lesssim_i^G$ for each i, if $(j, k) \in \lesssim_i^F$ then $(j, k) \in \lesssim_i^G$. □

Corollary 4.2.1. If $j \sim_i^F k$ for any i, j, k according to some comparative similarity assignment $(S_i^F, \lesssim_i^F)$ on a frame F, then $j \sim_i^G k$ according to any dilution G of F.

Proof. Immediate from Lemma 4.2 and definition of $\sim_i$. □

The dual relationship between frame refinements and frame dilutions, although implicit in the definition, deserves highlighting.

Lemma 4.3. For any ordering frames $F, G \in \mathbf{CS}$, $(F, G) \in \mathcal{R}$ iff $(F, G) \in \mathcal{D}$.

P r o o f. It is immediate from definitions of refinements and dilutions. □

Lemma 4.4. For any ordering frames $F = (W^F, \lesssim^F)$, $G = (W^G, \lesssim^G)$, and any V:

If $W^F = W^G$ and $A \in For_0$, then $(F, V), i \Vdash A$ iff $(G, V), i \Vdash A$.

P r o o f. It suffices to observe that the truth of formulae in For_0 is independent of $\lesssim$. □

2.1.5. Semantic properties of refinements and dilutions.

The following result is central to the key applications of the modified account. It is difficult to overstate its importance. It is the main result of ordering semantics for counterfactuals presented in this article. Refinements are *truth-preserving* in the following sense:

Proposition 5.1. If a counterfactual $A > B$ (such that A, $B \in For_0$) is *true* at a world according to some ordering frame F, then it is true at that world according to all refinements of F. That is, for all $F = (W^F, \lesssim^F) \in \mathbf{CS}$, and for all A, $B \in For_0$, $i \in W^F$, and V:

$$(F, V), i \Vdash A > B \text{ iff } (\forall G \in \mathcal{R}[F])((G, V), i \Vdash A > B)$$

P r o o f. ($\Longleftarrow$) Is immediate, since $F \in \mathcal{R}[F]$. ($\Longrightarrow$) Consider some $F \in \mathbf{CS}$, $A \in For_0$, $i \in W^F$, V such that $(F, V), i \Vdash A > B$. Hence, for all A, $B \in For_0$, $i \in W^F$, V either $\sim\exists k \in S_i^F$: $(F, V), k \Vdash A$ or $\exists k \in S_i^F$: $(F, V), k \Vdash A$ and $\forall j \in S_i^F$ $(j \lesssim_i^F k \longrightarrow (F, V), j \Vdash A \supset B)$. Let us start with the vacuous case and assume for arbitrary $A \in For_0$, $i \in W^F$, and V that $\sim\exists k \in S_i^F$: $(F, V), k \Vdash A$. From this, Lemma 4.4, and the fact that $S_i^G = S_i^F$ we can infer that $\sim\exists k \in S_i^G$: $(G, V), k \Vdash A$. Next, let us assume that $\exists k \in S_i^F$: $(F, V), k \Vdash A$ and $\forall j \in S_i^F$ $(j \lesssim_i^F k \longrightarrow (F, V), j \Vdash A \supset B)$. To distinguish it from other assumptions call this assumption *the main hypothesis*. It follows that $\exists k \in S_i^G$ and $(G, V), k \Vdash A$ for all $G \in \mathcal{R}[F]$, by Lemma 4.4 and the fact that $S_i^G = S_i^F$. Now, to show that $\forall j \in S_i^G$ $(j \lesssim_i^G k \longrightarrow (G, V), j \Vdash A \supset B)$ we will proceed by assuming $j \lesssim_i^G k$ for arbitrary $j \in S_i^G$, $G \in \mathcal{R}[F]$, and show $(G, V), j \Vdash A \supset B$. So, let us assume $j \lesssim_i^G k$ for arbitrary $j \in S_i^G$, $G \in \mathcal{R}[F]$, and note that since G is a refinement of F, then F is a dilution of G, by Lemma 4.3. Also, it should be noted that dilutions are $\lesssim$-preserving in the sense of Lemma 4.2. Hence, we conclude $j \lesssim_i^F k$, by Lemma 4.2 and Lemma 4.3. From this and *the main hypothesis*, we infer $(F, V), j \Vdash A \supset B$, which in conjunction with the fact that $W^F = W^G$ gives $(G, V), j \Vdash A \supset B$, by Lemma 4.4. Therefore, we finally conclude that $\forall j \in S_i^G$ $(j \lesssim_i^G k \longrightarrow (G,V), j \Vdash A \supset B)$, by conditional proof. This completes the proof. [16] □

We have a dual result for dilutions, which are *falsity-preserving* in the following sense:

Corollary 5.2. For all frames $F, G \in$ **CS** and for all $A, B \in For_0$, and V:

$$(G, F) \in \mathcal{D} \longrightarrow \forall i \in W^G)((G,V), i \nVdash A > B \longrightarrow (F, V), i \nVdash A > B)$$

Proof. We have the following from Proposition 5.1, for all $F, G \in$ **CS**, $A, B \in For_0$, and V:

1. $(F, G) \in \mathcal{R} \longrightarrow (\forall i \in W^F)((F,V), i \Vdash A > B \longrightarrow (G, V), i \Vdash A > B)$

Contraposing the consequent yields:

2. $(F, G) \in \mathcal{R} \longrightarrow (\forall i \in W^F)((G,V), i \nVdash A > B \longrightarrow (F, V), i \nVdash A > B)$

Finally, we obtain 3 by substituting an equivalent term in the antecedent of 2, by Lemma 4.3,

3. $(G, F) \in \mathcal{D} \longrightarrow (\forall i \in W^G)((G,V), i \nVdash A > B \longrightarrow (F, V), i \nVdash A > B)$

and note that whenever the antecedents of 2 and 3 are true, then $W^F = W^G$ is true, and the consequents of 2 and 3 are identical. If the antecedents of 2 and 3 are false, then both 2 and 3 are vacuously true, so the quantifier change is justified. □

2.1.6. Interpretation: contextual information.

Ordering frames, which constitute the basis of **CS** model theory are—much like systems of spheres—a means of carrying information about the comparative similarity of worlds, relative to any other world where a counterfactual's truth is being evaluated. On Lewis' (1981, §2) conception of comparative similarity, ordering frames, being largely determined by contextual considerations are to be viewed as *carriers of contextual information*.[17]

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[16] Lewis (1981, pp. 226–227) has proven a very similar result. His result is more general than Proposition 5.1 in one sense, and less general in another. Whereas Proposition 5.1 holds only for a class of frames based on *total preorderings*, Lewis has proven a similar result for ordering frames based on *partial orderings* (where only refinements are required to be based on total preorderings). On the other hand, whereas Lewis has proven this only for (strongly) *centered* ordering frames, Proposition 5.1 holds for *weakly centered* orderings frames, i.e., satisfying (CS3), so *a fortiori* it holds for ordering frames satisfying the (stronger) *centering* restriction (CS3.1). Also, the employment of frame dilutions and Lemmas 4.2 and 4.3. makes the proof of Proposition 5.1 substantially simper than Lewis' proof.

[17] Following (Lewis, 1973, §2.3; 1981, §2) in that regard.

> The ordering that gives the factual background depends on the facts about the world, known or unknown; how it depends on them is determined—or underdetermined—by our linguistic practice and by context. We may separate the contribution of practice and context from the contribution of the world, evaluating counterfactuals as true or false at a world, and according to a frame determined somehow by practice and context. (Lewis, 1981, p. 218)

Refinements, whilst containing more contextual information (when we refine, we add contextual information by making additional distinctions), preserve the contextual information of the original ordering frame. Another way of looking at this is to view those distinctions (absent from the original ordering frame) as becoming relevant on the context represented by the refinement. Dilutions do the opposite—they remove previously existing distinctions, so when we dilute we are removing contextual information (irrelevant information), i.e., distinctions that have been relevant on the context represented by the original frame are no longer relevant on the context represented by its dilution.

Usually we tend to think of submodels as providing less information than their extensions. But in this case, there is a sense in which the opposite seems to be happening. When we refine, we are taking submodels, and we can keep going until we get to a linear ordering: that direction feels like we are adding information. On the other hand, if we take supermodels (dilute), the limit is the case where everything is related to everything else, which feels like we are losing information. This tends to go against the usual intuitions.[18]

2.2. The Modified Account

2.2.1. Introduction.

The following sections constitute the model theory of the proposed analysis of contextualized counterfactuals, consisting of context representation, a formal language and its semantics. Setting up the basics of the semantics for the contextualized language, I designate (by way of proposal) the role of context representation to **CS** ordering frames (which constitute the basis of the **CS** account of counterfactuals) and argue that they are adequate for that purpose. The formal language for contextualized counterfactuals, introduces context-indexed connectives $>_c$ for each context c. That is, expressions like $A >_c B$ in the formal language intend to model contextualized counterfactuals of the form "In context c: If it were the case that A, then then it would be the case that B", where A and B express propositions. The corresponding semantics (**CS**+ model theory) of a language contextualized in that manner allows making distinctions in the truth value of counterfactuals with the same antecedents (and even the same antecedents and consequents), by appeal to contextual considerations explicitly indicated by their respective context indices.

[18] I owe this observation to Toby Meadows.

2.2.2. The modified language.

Each modified language is just like $\mathcal{L}$ given in Definition 1.1 that generates *For*, but instead of the single connective >, each contains a family of indexed connectives.

Definition 7.1. Let $\mathcal{L}^{\mathcal{C}} := \{\sim, \Box, \Diamond, \wedge, \vee, \supset\} \cup \{>_c : c \in \mathcal{C}\}$, where $\mathcal{C}$ is a set, regarded as a set of contexts.

The set of well-formed formulae $For^{\mathcal{C}}$ will reflect the intended analysis, so context-indices will not vary across nested $>_c$-formulae. I propose that the context-index of the main conditional connective $>_c$ of a nested conditional, e.g., $A >_c (B >_c C)$ should settle the matter of what information is imported into counterfactual worlds when evaluating its subformulae. I do this in Definition 7.3 by stipulating that nested indexed-conditionals inherit the context-index of the outermost indexed conditional.[19] The thought is that the information imported in evaluating the inner conditional is *contextually the same*, i.e., restricted by what information is imported in evaluating the outer conditional. But the information *is not the same simpliciter*, since the inner conditional need not have the same antecedent as the outer conditional, and its truth may not be evaluated at the same world as the outer conditional—both highly relevant factors that contribute to determining what information should be imported.

To define the set $For^{\mathcal{C}}$ of well-formed formulae of interest, it will be easier to first define a larger set, and subsequently apply the required (intended) restrictions.

Definition 7.2. Let $for^{\mathcal{C}}$ be the smallest set closed under the following well-formed formula formation rules:

B: All propositional variables are wffs, i.e., $PV \subseteq for^{\mathcal{C}}$.

R1: If $A \in for^{\mathcal{C}}$, then $\{\sim A, \Box A, \Diamond A\} \subseteq for^{\mathcal{C}}$.

R2: If $A, B \in for^{\mathcal{C}}$, then $\{A \wedge B, A \vee B, A \supset B\} \subseteq for^{\mathcal{C}}$.

R3: If $A, B \in for^{\mathcal{C}}$ and $c \in \mathcal{C}$, then $A >_c B \in for^{\mathcal{C}}$.

As mentioned earlier, indexed conditionals nested within other indexed conditionals inherit the indices of the outermost indexed conditional. It just does not make sense in this picture to speak of embedded conditionals whose indices vary. Below is the restriction on $for^{\mathcal{C}}$ that reflects this motivation.

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[19] The proposed approach may be interpreted as going some way of addressing a question posed by Priest (2018, Section 3.1, Endnote 14), regarding what information from the world where the counterfactual is evaluated should be imported into counterfactual worlds, when evaluating nested conditionals (counterfactuals).

Definition 7.3. Let $For^{\mathcal{C}}$ be the subset of $for^{\mathcal{C}}$ with the following restriction: for any single, nested formula $A >_c B$ where A or B contain instances of an indexed connective $>_x$ for some $x \in \mathcal{C}$, then $x = c$.

Example. Formulae such as $p >_a (q >_b r)$ or $(q >_b r) >_a p$, where $a \neq b$, are not elements of $For^{\mathcal{C}}$. However, the following are: $p >_a (q >_a r)$, $(q >_b r) >_b p$, $(p >_a q) \vee (r >_b s)$.

The following couple of definitions establish useful restrictions on $For^{\mathcal{C}}$, to which the key results will apply—namely unnested formulae. The following definition characterizes the part of $For^{\mathcal{C}}$ whose elements contain no nested indexed conditionals. If indexed conditionals exist, their antecedents and consequents are basic modal logic formulae.

Definition 7.4. Let $For^{\mathcal{C}}_{>0}$ be the subset of $For^{\mathcal{C}}$ such that for any formula of the form $A >_c B$, the following restriction applies: $A, B \in For_0$.

Example: $\sim(p >_a (q \supset r)) \wedge (((p \wedge \sim q) >_b r) \vee (q >_c r)) \in For^{\mathcal{C}}_{>0}$ for any a, b, $c \in \mathcal{C}$. But $p >_c (p >_c p) \notin For^{\mathcal{C}}_{>0}$ for no $c \in \mathcal{C}$.

The following definition characterizes the part of $For^{\mathcal{C}}$ whose elements have an indexed conditional connective as the main connective.

Definition 7.5. Define $For^{\mathcal{C}}(>) \coloneqq \{A >_c B: A, B \in For^{\mathcal{C}}, c \in \mathcal{C}\}$. That is, $For^{\mathcal{C}}(>)$ is just the set of $For^{\mathcal{C}}$ formulae whose main connective is $>_c$, for some $c \in \mathcal{C}$.

2.2.3. Modified model theory.

The semantics for the contextualized language draws heavily on **CS** model theory (intended to serve as the foundation for **CS**+ model theory) by developing a formalism that reduces the truth conditions for $A >_c B$ on a **CS**+ model to those for $A > B$ on a corresponding **CS** model whose underlying ordering frame is taken to represent context c. That is, contextual considerations underlying a context-indexed expression are cashed out in terms of contextual information carried by ordering frames. Some tentative suggestions to that effect can be found in Nolan's (1997, n. 28).

The formula $A >_c B$ is intended to be read as an explicitly contextualized version of $A > B$. That is, the model theory in this section gives an analysis of $A >_c B$, which is to be read as: "In context c: If it were the case that A, then it would be the case that B".

The following definition will play a key role in the defining the truth conditions for indexed counterfactuals, i.e., for the truth conditions of formulae like $A >_c B$.

Definition 7.8. Let $\underline{\ \ \ }$: $For^{\mathcal{C}} \longrightarrow For$ be the function that transforms all formulae with indexed connectives $>_c$ for any $c \in \mathcal{C}$ into unindexed ones $>$, in all subformulae of a formula. That is, it "strips" any $For^{\mathcal{C}}$ formula of its indices, leaving its index-less *For* counterpart.

B: $\underline{p} = p$ for all $c \in PV$.

R1: $\underline{*\,A} = *\,\underline{A}$ for each $* \in \{\sim, \Box, \Diamond\}$ and $A \in For^{\mathcal{C}}$.

R2: $\underline{A \circ B} = \underline{A} \circ \underline{B}$ for each $\circ \in \{\wedge, \vee, \supset\}$ and $A, B \in For^{\mathcal{C}}$.

R3: $\underline{A >_c B} = \underline{A} > \underline{B}$ for each $c \in \mathcal{C}$ and $A, B \in For^{\mathcal{C}}$.

Example. $\underline{\sim p >_c (q \vee r)} = \sim p > (q \vee r)$.

It will be useful to extend the above definition of the "index-elimination function" to sets of formulae. No ambiguity should arise whether the argument is a formula or a set of formulae.

Definition 7.9. For any $\Sigma \subseteq For^{\mathcal{C}}$, let $\underline{\Sigma} \coloneqq \{\underline{A} \in For \colon A \in \Sigma\}$.

Definition 8.1. A **CS**+ frame of the modified language is a triple $(W, \mathcal{C}, r)$, where $W \neq \emptyset$ and $\mathcal{C} \neq \emptyset$ are sets, and $r\colon W \times \mathcal{C} \longrightarrow \wp(W) \times \wp(W \times W)$ is a function such that $r(i, c)$ satisfies conditions CS1–CS5 for each $c \in \mathcal{C}$.[20]

On the intended interpretation W is regarded as a set of possible worlds, $\mathcal{C}$ is regarded as a set of contexts, and r is regarded a comparative similarity assignment to world i in context c, i.e., $r(i, c) = (S_{i,c}, \lesssim_{i,c})$ such that $S_{i,c}$ and $\lesssim_{i,c}$ satisfy conditions CS1–CS5. Informally, the set of all similarity assignments restricted to some index $c \in \mathcal{C}$, i.e., $\{r(i, c)\colon i \in W\} = r[W \times c]$ is regarded as *representing* (reflecting) a context indexed by c.[21]

Putting it another way, in line with Lewis, we could say that $r[W \times c]$ is the ordering frame determined somehow by practice and context c. It should be observed that collecting such assignments for all $i \in W$ and some fixed $c \in \mathcal{C}$ will give $\{r(i, c)\colon i \in W\} = \lesssim^F [W]$, where $F \in$ **CS**, i.e., a comparative similarity assignment that is identical to one given by some **CS** ordering frame. Hence, the image of r over all elements of $W \times \mathcal{C}$ would be identical to a collection of com-

[20] It should be noted that from a purely formal perspective, **CS**+ frames are essentially Lewisean in spirit. Not only does the domain of r (the extended counterpart to Lewisean \$ or $\lesssim$) satisfy the condition of the general formalism envisaged by Lewis (1973, p. 119) of being a set, but the image of r contains total preorders, which are just the Lewisean (1973; 1981) ordering semantics counterparts to systems of spheres.

[21] This is standard notation for the image of a set under some function, i.e., $r[W \times c] = \{r(i, c)\colon i \in W\}$, and similarly $\lesssim^F [W] = \{\lesssim^F_i\colon i \in W\}$.

parative similarity assignments given by a subset of **CS** frames. That is $r[W \times \mathcal{C}] \subseteq \{\lesssim^F [W]: F \in \textbf{CS}\}$. Informally and succinctly, we could say that each **CS**+ frame acts like a set of **CS** frames.

Some of the above informal observations can be made precise in the following lemma, which will have some important applications in developing a precise account of a contextualized consequence relation.

Lemma 8.1. For any $\mathfrak{F} = (W, \mathcal{C}, r) \in \textbf{CS}+$ and any $c \in \mathcal{C}$ there exists a unique ordering frame $F = (W, \lesssim^F) \in \textbf{CS}$ such that $r(i, c) = (S_i^F, \lesssim_i^F)$ for each $i \in W$, or equivalently the following holds: $\{r(i, c): i \in W\} = r[W \times c] = \lesssim^F [W]$. Also, for any $F = (W, \lesssim^F) \in \textbf{CS}$ there exists a unique family $\mathbb{F}$ of **CS**+ frames whose assignments for some fixed $c \in \mathcal{C}$ and all worlds are exactly the comparative similarity assignments of F. Formally, this family would be $\mathbb{F} = \{(W, \mathcal{C}, r) \in \textbf{CS}+: \exists c \in \mathcal{C}(\{r(i, c): i \in W\} = \lesssim^F [W])\}$ for some $F \in \textbf{CS}$.

P r o o f. Observe that each $r(i, c)$ satisfies CS1–CS5 by definition, making $F = (W, \lesssim^F) \in \textbf{CS}$, as required. Moreover, F is unique since $W^F = W^{\mathfrak{F}}$. To see that there exists a unique family of **CS**+ frames for any $F \in \textbf{CS}$, it suffices to see that $W^F = W^{\mathfrak{F}}$ and that $\lesssim^F [W]$ is just a collection of comparative similarity assignments and if some $\mathfrak{F} = (W, \mathcal{C}, r) \in \textbf{CS}+$ contains a context index $c \in \mathcal{C}$ such that $\{r(i, c): i \in W\} = \lesssim^F [W]$, then $\mathfrak{F} \in \mathbb{F}$, else $\mathfrak{F} \notin \mathbb{F}$. □

Given the above result we can establish some useful notation, which will be crucial in giving a succinct expression of contextualized validity as well as an important theorem.

Definition 8.2. Given *Lemma 8.1*, and given a $\mathfrak{F} = (W, \mathcal{C}, r) \in \textbf{CS}+$, and any $c \in \mathcal{C}$, denote with $F_{\mathfrak{F}}(c)$ the unique $F = (W, \lesssim^F) \in \textbf{CS}$ such that $r[W \times c] = \lesssim^F [W]$.

The motivation for the following definition stems from expressing contextualized validity as succinctly and clearly as possible. Here we introduce notation that bridges semantic notions such as ordering frames and ordering frame refinements with the corresponding syntactic notions of context indices, explicitly present in the formal language. This notation will be key in the formulation of Theorem 8.6 and Definition 9.1 (contextualized consequence relation).

Definition 8.3. For any $\mathfrak{F} = (W, \mathcal{C}, r) \in \textbf{CS}+$ and $a, b \in \mathcal{C}$ let $b \leq a$ iff $(F_{\mathfrak{F}}(a), F_{\mathfrak{F}}(b)) \in \mathcal{R}$. That is, $b \leq a$ iff $F_{\mathfrak{F}}(b)$ is a refinement of $F_{\mathfrak{F}}(a)$.

Definition 8.4. A **CS**+ model of the modified language is the quadruple:

$$(W, \mathcal{C}, r, V)$$

where $(W, \mathcal{C}, r)$ is a **CS**+ frame and V is as in Definition 2.6.

Definition 8.4.1. Truth in **CS**+ models is defined via a satisfiability relation $\Vdash^{\mathcal{C}} \subseteq W \times For^{\mathcal{C}}$. We read $i \Vdash^{\mathcal{C}} A$ as "A is true at i". Given a **CS**+ model $(W, \mathcal{C}, r, V)$ and any $i \in W$, define $\Vdash^{\mathcal{C}}$ as follows:

(1) $i \Vdash^{\mathcal{C}} p$ iff $V_i(p) = 1$
(2) $i \Vdash^{\mathcal{C}} {\sim}A$ iff not $i \Vdash^{\mathcal{C}} A$
(3) $i \Vdash^{\mathcal{C}} A \wedge B$ iff $i \Vdash^{\mathcal{C}} A$ and $i \Vdash^{\mathcal{C}} B$
(4) $i \Vdash^{\mathcal{C}} A \vee B$ iff $i \Vdash^{\mathcal{C}} A$ or $i \Vdash^{\mathcal{C}} B$
(5) $i \Vdash^{\mathcal{C}} A \supset B$ iff $i \Vdash^{\mathcal{C}} {\sim}A$ or $i \Vdash^{\mathcal{C}} B$
(6) $i \Vdash^{\mathcal{C}} \Box A$ iff $\forall j \in W$: $j \Vdash^{\mathcal{C}} A$
(7) $i \Vdash^{\mathcal{C}} \Diamond A$ iff $\exists j \in W$: $j \Vdash^{\mathcal{C}} A$
(8) $i \Vdash^{\mathcal{C}} A >_c B$ iff ${\sim}\exists k \in S_{i,c}$: $k \Vdash \underline{A}$, or
$\exists k \in S_{i,c}$: $k \Vdash \underline{A}$ and $\forall j \in S_{i,c}(j \lesssim_{i,c} k \longrightarrow j \Vdash \underline{A \supset B})$

Note that if this definition is restricted to $For^{\mathcal{C}}_{>0}$, as indeed many of our results are, then resorting to the index-elimination function is unnecessary, and we can just give the following, simpler expression:

(8') $i \Vdash^{\mathcal{C}} A >_c B$ iff ${\sim}\exists k \in S_{i,c}$: $k \Vdash A$, or
$\exists k \in S_{i,c}$: $k \Vdash A$ and $\forall j \in S_{i,c}(j \lesssim_{i,c} k \longrightarrow j \Vdash A \supset B)$

We have an analogous result to Proposition 5.1 for **CS**+ models, if we observe, as Lemma 8.1 shows, that **CS** models are embedded within **CS**+ models. Let us consider a relationship much like refinements but defined between collections of comparative similarity assignments for some fixed $c \in \mathcal{C}$, and all $i \in W$, i.e., $r[W \times c]$, which Lemma 8.1 shows to be identical to **CS** ordering frames. That is, let us extend the notion of refinements to **CS**+ frames as follows.

Definition 3.1.1. For any **CS**+ frame $(W, \mathcal{C}, r)$ call $r[W \times b]$ a *refinement* of $r[W \times a]$ iff for all $i \in W$:

(i) $\lesssim_{i,b} \subseteq \lesssim_{i,a}$
(ii) $S_{i,a} = S_{i,b}$

Note that condition (i) of domain identity in Definition 3.1 of refinements on **CS** frames on this definition is automatically satisfied, since it is defined on a single **CS**+ frame. Let us abbreviate $r[W \times a]$ with r_a for any **CS**+ frame. We could borrow the notation $(r_a, r_b) \in \mathcal{R}$ to say that r_b is a *refinement* of r_a. Now we get the **CS**+ counterpart of Proposition 5.1 for free.

Corollary 5.1.1. If a counterfactual $A >_a B$ (such that $A, B \in For_0$) is true at some world and r_b is a refinement of r_b, then $A >_b B$ true at that world. That is, for all $\mathfrak{A} = (W, \mathcal{C}, r, V)$, $A, B \in For_0$, $i \in W$, $a, b \in \mathcal{C}$, and V:

$$\mathfrak{A}, i \Vdash^{\mathcal{C}} A >_a B \quad \text{iff} \quad (\forall r_b \in \mathcal{R}[r_a])(\,\mathfrak{A}, i \Vdash^{\mathcal{C}} A >_b B)$$

P r o o f. Each r_c for any $c \in \mathcal{C}$ has the properties of a **CS** ordering frame, by Lemma 8.1, so the result follows by Proposition 5.1. □

Let us introduce further notation that will make subsequent, key expressions more succinct.

Definition 8.5. As in the case of **CS** models, let us introduce the following notation for convenience: $\mathfrak{A}$, $i \Vdash^{\mathcal{C}} \Sigma$ iff $\mathfrak{A}$, $i \Vdash^{\mathcal{C}} A$ for all $A \in \Sigma$. Also denote with $\mathfrak{A} \Vdash^{\mathcal{C}} A$ when $\mathfrak{A}$, $i \Vdash^{\mathcal{C}} A$ for all $i \in W^{\mathfrak{A}}$.

Just as we have relativized *formula validity* to a model $\mathfrak{A} \Vdash^{\mathcal{C}} A$ in the definition above, it will be of use to define *valid inference* relativized to a model.

Definition 8.6. Let $\vDash^{\mathcal{C}}_{\mathfrak{A}} \subseteq \wp(For^{\mathcal{C}}) \times For^{\mathcal{C}}$, and given a **CS**+ model $\mathfrak{A} = (W, \mathcal{C}, r, V)$, write

$\vDash^{\mathcal{C}}_{\mathfrak{A}} A$	iff	$\mathfrak{A} \Vdash^{\mathcal{C}} A$,
$\Sigma \vDash^{\mathcal{C}}_{\mathfrak{A}} A$	iff	for all $i \in W$: if $\mathfrak{A} \Vdash^{\mathcal{C}} \Sigma$, then $\mathfrak{A} \Vdash^{\mathcal{C}} A$.

Now we proceed to define formula validity and semantic consequence of the contextualized language on the proposed **CS**+ model theory.

Definition 8.7 (**CS**+ validity)**.** Define the relation $\vDash^{\mathcal{C}}_{\mathbf{CS+}} \subseteq \wp(For^{\mathcal{C}}) \times For^{\mathcal{C}}$, as follows: $\Sigma \vDash^{\mathcal{C}}_{\mathbf{CS+}} A$ iff for all **CS**+ models $\mathfrak{A}$ and $i \in W$: if $\mathfrak{A}$, $i \Vdash^{\mathcal{C}} \Sigma$, then $\mathfrak{A}$, $i \Vdash^{\mathcal{C}} A$.

We say an inference from Σ to A is **CS**+ valid iff $\Sigma \vDash^{\mathcal{C}}_{\mathbf{CS+}} A$. That is, valid inference is defined as truth preservation at all worlds in all **CS**+ models. A formula $A \in For^{\mathcal{C}}$ is said to be **CS**+ valid iff $\emptyset \vDash^{\mathcal{C}}_{\mathbf{CS+}} A$. Notation from Definition 8.6 allows us to express the **CS**+ semantic consequence more succinctly: $\Sigma \vDash^{\mathcal{C}}_{\mathbf{CS+}} A$ iff for all **CS+** models $\mathfrak{A}$: $\Sigma \vDash^{\mathcal{C}}_{\mathrm{A}} A$.

Note that it is immediate from the above definitions that $\vDash^{\mathcal{C}}_{\mathbf{CS+}} \subseteq \vDash^{\mathcal{C}}_{\mathfrak{A}}$ for any **CS**+ model $\mathfrak{A}$.

It should be also noted that since the truth conditions for □ and ◊ formulae are defined in terms of unrestricted quantification over possible worlds, i.e., only $>_c$-formulae truth conditions depend on $\mathcal{C}$ and r, the above validity conditions give the modal logic **S5** for the basic modal language.

The part of the basic modal language is indistinguishable between the two classes of models in the following sense.

Lemma 8.2. If for any **CS** model $\mathfrak{A}$, **CS**+ model $\mathfrak{B}$ such that $W^{\mathfrak{A}} = W^{\mathfrak{B}}$ and $V^{\mathfrak{A}} = V^{\mathfrak{B}}$, then for any $A \in For_0$, and $i \in W$: $\mathfrak{A}, i \Vdash A$ iff $\mathfrak{B}, i \Vdash^{\mathcal{C}} A$.

Proof. It suffices to note that elements of For_0 depend only on W and V. □

That is, the classes of **CS** models and **CS**+ models validate exactly the same formulae of For_0.

Theorem 8.3. If $\Sigma \cup \{A\} \subseteq For_0$: then $\Sigma \vDash_{\mathbf{CS}} A$ iff $\Sigma \vDash_{\mathbf{CS+}} A$.

Proof. Immediate from Lemma 8.2 and the definitions of $\vDash_{\mathbf{CS}}$ and of $\vDash_{\mathbf{CS+}}$. □

2.2.4. Main results of the modified account.

Aside from the formulation of the contextualized consequence relation given in the next section, Theorem 8.5 and Theorem 8.6, formulated and proved in this section are the main results of the modified account. Lemma 8.2 and Theorem 8.3 sanction Theorem 8.5, which captures our intuition regarding the contextualized language—if we restrict our discourse to a single context on any occasion, then we should expect **CS**+ analysis (indexed account) reduce to the **CS** analysis (unindexed account).

The second of the two main theorems, Theorem 8.6—sanctioned by the application of Proposition 5.1 in a key step of its proof—states that part of the logic given by **CS** semantic consequence relation can be preserved on **CS**+ models if the conclusion context preserves the contextual information of the contexts over which the premises range.[22]

Definition 8.8. Call frame $H \in \mathbf{CS}$ a *mutual refinement* of frames F and G iff $(F, H) \in \mathcal{R}$ and $(G, H) \in \mathcal{R}$. Note that H is a mutual refinement of F and G iff $H \in \mathcal{R}[F] \cap \mathcal{R}[G]$.[23]

[22] It is worthwhile following the proof of Theorem 8.6 to see how other results are employed, but in particular how Proposition 5.1 is applied in securing contextual information preservation—from the premises to the conclusion—as that should may offer insight to understanding the formulation of contextualized validity.

[23] For a reminder of the meaning of $\mathcal{R}$ and $\mathcal{R}[F]$, see Definitions 3.1 and 3.1.2, respectively.

There is another important fact, that goes beyond For_0 that we will need in proving Theorems 8.5 and 8.6. Informally speaking it states that **CS**+ models behave much like collections of **CS** models. That is, whenever we restrict our discourse to a single context, modelled by **CS**+ models restricted to some single context index, there is a **CS** model that gives us the same analysis. This result and subsequently Theorem 8.5 can be viewed as a formal vindication of the objection expressed in Section 1, that analyses on the contested class are already restricted in that manner. The following lemma establishes the above informal observation.

Lemma 8.4. Given $(\mathfrak{F}, V) \in$ **CS**+ and $(F_{\mathfrak{F}}(c), V)$, for any $c \in \mathcal{C}$ and any $A >_c B \in For^{\mathcal{C}}$:

$$(\mathfrak{F}, V), i \Vdash^{\mathcal{C}} A >_c B \text{ iff } (F_{\mathfrak{F}}(c), V), i \Vdash \underline{A >_c B}$$

P r o o f. The result follows directly from the definition of $F_{\mathfrak{F}}(c)$ and **CS**+ truth conditions for indexed formulae. That is, $(\mathfrak{F}, V), i \Vdash^{\mathcal{C}} A >_c B$ is given in terms of $\underline{A >_c B}$ being true according to the comparative similarity assignment $r(i, c)$, but the comparative similarity assignment at i on $F_{\mathfrak{F}}(c)$ is just $r(i, c)$ by Definition 8.2. □

Definition 8.9. Denote $For^{\mathcal{C}}_{>0} \cap For^{\mathcal{C}}(>)$ with $For^{\mathcal{C}}_{>0}(>)$.[24]

Definition 8.10. Let Ind: $\wp(For^{\mathcal{C}}) \longrightarrow \wp(\mathcal{C})$ be the function that outputs the set of all indices appearing in a set of formulae; e.g., $Ind(\{p >_c q\}) = \{c\}$, $Ind(\{p >_a q, p >_b q\}) = \{a, b\}$.

All valid **CS** inference patterns are preserved on the modified account, whenever the premises and conclusions range over at most a single context index. This makes sense intuitively, and the semantics manages to align with our intuition in this regard. This can almost be stated without proof, as a corollary of Lemma 8.4, but I provide one anyway, only if to highlight some important relationships between **CS** and **CS**+ models.

Theorem 8.5. For all $\Sigma \cup \{A\} \subseteq For^{\mathcal{C}}$: If $\underline{\Sigma} \vDash_{\mathbf{CS}} \underline{A}$ and $|Ind(\Sigma \cup \{A\})| \leq 1$, then $\Sigma \vDash_{\mathbf{CS+}} A$.

In other words, if the unindexed inference is **CS** valid, and if the premises and conclusion range over at most one context-index, then the inference is **CS**+ valid.

P r o o f. An informal argument should suffice, if we observe that restricting the inference to at most a single context index, effectively restricts the analysis to

[24] See Definition 7.4 and Definition 7.5.

a basic modal language with a single indexed conditional connective $>_c$. That is, with Lemma 8.4 and Theorem 8.3 it can be shown that such a restriction makes **CS**+ models behave exactly like **CS** models. □

E x a m p l e. For all $A, B \in For^{\mathcal{C}}$, and all $c \in \mathcal{C}$:

$\vDash_{\mathbf{CS}+} A >_c A$

$A, A >_c B \vDash_{\mathbf{CS}+} B$

$\Box(A \supset B) \vDash_{\mathbf{CS}+} A >_c B$

$\vDash_{\mathbf{CS}+} {\sim}((A >_c B) \wedge (A >_c {\sim}B))$

An important generalization of Theorem 8.5 can be given by employing truth preserving properties of ordering frame refinements, given by Proposition 5.1. This generalization is a step toward establishing a notion of contextualized validity, to which we shall turn our attention to in the next section.[25]

Theorem 8.6. For all $\Sigma \cup \{A\} \subseteq For^{\mathcal{C}}_{>0}(>) \cup For_0$:

If (1) $\underline{\Sigma} \vDash_{\mathbf{CS}} \underline{A}$ and

(2) if $Ind(\{A\}) = \{a\}$, then $a \leq b$ for all $b \in Ind(\Sigma)$ for each **CS**+ frame,

then $\Sigma \vDash_{\mathbf{CS}+} A$.

In other words, if the unindexed inference is **CS** valid and the conclusion index corresponds to an ordering frame that is a refinement of all ordering frames that correspond to the indices over which the premises range, then the inference is also **CS**+ valid. We interpret condition (2) as saying that the context on which the conclusion is evaluated is not independent of the contexts on which the premises are evaluated. That is, the conclusion context is supposed to *preserve the contextual information* carried by contexts on which the premises are evaluated. It is hoped that the following, informal proof will be insightful.

P r o o f. Let $(\mathfrak{F}, V)$ be a **CS**+ model where all $B \in \Sigma$ are true at some world i. Now, each $\underline{B} \in \underline{\Sigma}$ originally indexed by $b \in Ind(\Sigma)$ is also true at i according to $(F_{\mathfrak{F}}(b), V)$ by Lemma 8.4. Next, given that $(F_{\mathfrak{F}}(a), V)$ is a *mutual refinement* of

[25] Note that the restriction to $For^{\mathcal{C}}_{>0}(>)$ stems from the fact that ordering frame refinements are only truth preserving, and that is the part of $For^{\mathcal{C}}$ to which Proposition 5.1 applies. Just to be clear, if $A >_c B \in For^{\mathcal{C}}_{>0}(>)$, then $A, B \in For_0$. That is, $A \in For^{\mathcal{C}}_{>0}(>)$ iff $|Ind(\{A\})| = 1$. In other words, this result apples to a language restricted to the basic propositional modal language with indexed conditionals appearing only as the main connectives to formulae that do not contain any other indexed conditionals as proper subformulae. I have stated this in the overview of the current section.

each ($F_{\mathfrak{F}}(b)$, V) by condition (2), Proposition 5.1 grants that each $\underline{B} \in \underline{\Sigma}$ is also true at i according to ($F_{\mathfrak{F}}(a)$, V). Therefore, $\underline{A}$ is also true at i according to ($F_{\mathfrak{F}}(a)$, V), since (1) is assumed. Finally, we see that A is also true at i according to ($\mathfrak{F}$, V), by Lemma 8.4. Hence, the inference is **CS**+ valid, as required. □

2.2.5. Contextualized validity.

We close the discussion by giving the definition of contextualized validity, and show that it fares well with inference patterns that motivated this account. **CS**+ is very weak since on the current definition of **CS**+ validity via **CS**+ semantic consequence relation, there are no conditions placed on the relationship between context-indices appearing in the premises and the conclusion. But this is inadequate if we wish to fashion a logic that is sensitive to explicit contextual content. That is, we have developed an analysis of the contextualized language but have only included *truth preserving* conditions for validity in that definition—naturally, we also want a notion of *contextual information preserving* conditions on the new, contextualized notion of valid inference. That is, currently, by Definition 8.4 we have the following condition for **CS**+ valid inference:

$$\Sigma \vDash^{\mathcal{C}}_{\mathbf{CS}+} A \text{ iff } \Sigma \vDash^{\mathcal{C}}_{\mathfrak{A}} A \text{ for all } \mathbf{CS}+ \text{ models } \mathfrak{A}.$$

Clearly, these validity conditions are no different from those for **CS** and as such inadequate for the notion of a consequence relation that takes into account relationships that may exist between the context indices of formulae in the premises and conclusion. Consequently, such conditions make **CS**+ unacceptably weak, because for every **CS** valid inference there will be a counterexample by choice of indices for the premises and conclusion such that the premises are true, and the conclusion is false.

Theorem 8.6 captures some of the contextual information preserving features that hint at how contextual constraints could be fashioned. The theorem tells us that if we restrict the language in a way that Proposition 5.1 can be implemented, then **CS** validity and valid inference is preserved if additional conditions on the relationship between the premise indices and conclusion index are satisfied, i.e., conditions that correspond to what we mean by contextual information preservation. This opens a possibility for defining a notion of valid inference that those conditions underlie. That is, we could fashion a notion of contextualized inference by adding condition (2) of Theorem 8.6 to the current definition of **CS**+ valid consequence. The key definition that requires modification is of $\Sigma \vDash^{\mathcal{C}}_{\mathfrak{A}} A$, defined (below) since $\Sigma \vDash^{\mathcal{C}}_{\mathbf{CS}+} A$ is defined in terms of it.

Definitions 9.1 and 9.2 establish a proper logic of contextualized counterfactuals. That is, a logic where valid inference is not defined merely in terms of truth preservation but also in terms of *contextual information preservation*.

Definition 9.1. For a **CS**+ model $\mathfrak{A}$ let $\vDash^{\mathcal{C}}_{\mathfrak{A}} \subseteq \wp(For^{\mathcal{C}}_{>0}(>) \cup For_0) \times For^{\mathcal{C}}_{>0}(>) \cup For_0$, be defined as: $\Sigma \vDash^{\mathcal{C}}_{\mathfrak{A}} A$ iff

(1) $\forall i \in \mathfrak{A}[i \Vdash^{\mathcal{C}} \Sigma \longrightarrow i \Vdash^{\mathcal{C}} A]$ and

(2) $\forall F \in$ **CS**+ $[Ind(\Sigma) \neq \emptyset \longrightarrow \exists a \in Ind(\{A\})\ \forall b \in Ind(\Sigma)[a \leq b]]$

Condition (1) demands *truth preservation* at all worlds in a model whereas condition (2) requires *contextual information preservation* at all worlds in a model, making it the uniquely characteristic feature of the proposed, contextualized account. The requirement in condition (2) that for each model there must exist a mutual refinement of all the premise contexts intends to capture the idea of the necessary condition of there being a context that preserves some of the contextual information that is present in the premises. If there is no such context, then it makes little sense to speak of the conclusion being true anywhere else.[26]

Definition 9.2. Now we can finally define contextualized validity as follows:

$$\Sigma \vDash^{\mathcal{C}}_{\mathbf{CS+}} A \text{ iff } \Sigma \vDash^{\mathcal{C}}_{\mathfrak{A}} A \text{ for all } \mathbf{CS+} \text{ models } \mathfrak{A}.$$

Part of the motivation for the contextualized account was to invalidate inferring $A > (B \wedge C)$ from $A > B$ and $A > C$, due to some obvious counterexamples. We know that it is **CS** valid and on other conditional logics, because for the kind of instances that we view as worrisome, the premises can never be true, and as such go through vacuously.

3. Advantages of the Modified Account

3.1. Accounting for Contextual Differences

The proposed analysis overcomes all the shortcomings of the analyses that have been identified in Section 1. We can (i) have models that allow the evaluation both elements in each pair (1)–(2) and (3)–(4) as true at a single world. Moreover, (ii) in each case, such evaluation does not commit the analysis to the truth of the conditional with the same antecedent and the conjunction of the consequents of the conditionals in each pair. That is, *Adjunction of Consequents* is not valid if context indices that range over the premises are allowed to vary, i.e.,

[26] It may be worthwhile sharing the observation that the requirement of the existence of a context index that is said to preserve some information mutual to all premise indices (present in condition (2) of Definition 9.1) resembles in its form the syntactic, propositional variable sharing condition for valid relevant conditionals in the definition of relevant logic validity: "A propositional logic is relevant iff whenever $A \longrightarrow B$ (where '$\longrightarrow$' denotes logical implication) is logically valid, A and B have a propositional variable in common" (Priest, 2008, Section 9.7.8).

we can never infer $A >_c (B \wedge {\sim}B)$, and for similar reasons we need not commit to (5) from the truth of (1) and (2). Moreover, Theorem 8.5 guarantees that the analysis does not invalidate ${\sim}((A >_c B) \wedge (A >_c {\sim}B))$, i.e., the principle of *Conditional Non-Contradiction* for any context variable c (that principle is **CS**+ valid).

Example. To illustrate both (i) and (ii), let us consider (3) and (4) once more:

3. If I was Julius Caesar, I (Caesar) would not be alive in the 21st century.
4. If I was Julius Caesar, he (Caesar) would be alive in the 21st century

On the modified account we can evaluate both (3) and (4) as true, albeit relative to their contexts. This is done by reformulating them in the contextualized language with the enthymematic/nominal contextual content revealed, as follows:

3.1. In context a: If I was Julius Caesar, I (Caesar) would not be alive in the 21st century.

4.1. In context b: If I was Julius Caesar, he (Caesar) would be alive in the 21st century.

That is, (3) and (4) are effectively analysed as (3.1) and (4.1). It can be easily checked that it is possible to have both evaluated as true at a single world on **CS**+ models, since the set of relevant antecedent worlds (where I am Caesar) in context a is not the same as the set of relevant antecedent worlds (where I am Caesar) in context b. So, the joint truth of (3.1) and (4.1) does not force nor require the existence of relevant antecedent worlds where I (Caesar) am both alive and not alive in the 21st century.

To illustrate (ii), an informal argument will suffice, followed by a formal counterexample to *Adjunction of Consequents*. Let us formalize (3.1) and (4.1) by recourse to $For^{\mathcal{C}}$ and denote (3.1) with $A >_a B$ and (4.1) with $A >_b C$. Now it will be shown that $A >_c (B \wedge C)$ need not follow from $A >_a B$ and $A >_b C$, which is certainly desired. Although on the contextualized analysis there are now contexts a and b such that both premises $A >_a B$ and $A >_b C$ can be evaluated as true at some possible world i, there is no context c such that $c \leq a$ and $c \leq b$. In other words, it is not possible to integrate the contextual information of contexts a and b, carried by the corresponding comparative similarity ordering assignments $r(i, a)$ and $r(i, b)$ in a manner that corresponds to some possible context c whose information would be carried by the comparative similarity ordering assignment $r(i, c)$. The following counterexample to *Adjunction of Consequents*, where the consequent is an explicit contradiction, formally spells out the above informal argument.

Proposition 9.1. $p >_a q, p >_b {\sim}q \nvDash^{\mathcal{C}}_{\mathbf{CS+}} p >_c (q \wedge {\sim}q)$.

P r o o f. It suffices to provide a countermodel. Let $\mathfrak{A} = (W, \mathcal{C}, r, V)$ be a **CS**+ model as follows:

$$W = \{i, j, k\}, \mathcal{C} = \{a, b, c\}$$

Below, in the characterization of $\lesssim_{i,a}$ and $\lesssim_{i,b}$ the ellipses indicate the reflexive cases.

$$r(i, a) = (\{i, j, k\}, \{(i, j), (i, k), (j, k), \ldots\})$$
$$r(i, b) = (\{i, j, k\}, \{(i, k), (i, j), (k, j), \ldots\})$$
$$V_i(p) = 0$$
$$V_j(p) = 1 \qquad V_j(q) = 1$$
$$V_k(p) = 1 \qquad V_k(q) = 0$$

It is easy to check that both $i \Vdash^{\mathcal{C}} p >_a q$ and $i \Vdash^{\mathcal{C}} p >_b {\sim}q$ and that there is no ordering assignment $r(i, c)$ corresponding to index c that would be a mutual refinement of both $r(i, a)$ and $r(i, b)$. In particular there is no $\lesssim_{i,c}$ such that both $\lesssim_{i,c} \subseteq \lesssim_{i,a}$ and $\lesssim_{i,c} \subseteq \lesssim_{i,b}$. The only mutual information that $\lesssim_{i,a}$ and $\lesssim_{i,b}$ share is $\{(i, i), (j, j), (k, k)\}$, which fails to be a similarity assignment, since it is not total. Hence, the existence requirement of condition (2) of Definition 9.2 of **CS**+ contextualized validity is not satisfied. Hence, the above is a counterexample to *Adjunction of Consequents*, as required. □

What is paradigmatic about those counterexamples is that they highlight precisely what is really at play in contextualized validity when we explore limit cases, i.e., where the premises are true in radically different contexts (up to inconsistency). That is, we can have possible premises true for any contexts, but the inference is valid only if the conclusion can always be true in a contextually meaningful way—one that is not independent of the contextual information by virtue of which the premises are true. If there is no mutual refinement of comparative similarity assignments that represent context-indices over which the premises range, then there is no contextually meaningful way of speaking of the conclusion following from those premises. Therefore, the inference is contextually invalid. It should be noted that the inference fails in limit cases as exemplified in Proposition 9.1, but may very well go through on some **CS**+ models if the divergence of contexts over which the premises range is not completely incompatible, as the case may be with Quine's example of Caesar using both nuclear weapons and catapults. It could be argued that such contextual incompatibility of premises—all true but on contexts that do not have a mutual refinement—should be treated in the manner that inconsistent sets of premises are treated, i.e., the conclusion should follow vacuously. Perhaps this needs some more thought, but instances such as Proposition 9.1—which appear to be legitimate counterexamples to *Adjunction of Consequents*—seem to speak against such an approach (we want the inference to fail). The inference is invalid, and the contextualized account presented in this article gives the corresponding correct analysis.

3.2. A Note on Indicatives

The contextualized account lends itself to broader applications. It can be shown to serve as an explanatory device whilst offering a satisfactory analysis of a class of common phenomena, related to indicative conditionals, known as "Gibbardian Stand-Offs". The modified account fares better than the accounts in the contested class for the same reasons it did in the case of counterfactuals (subjunctive) conditionals. Indicative conditionals give rise to so called "stand-offs" when there are equally good reasons for two speakers to assert (on a single occasion) two conditionals that are in stark disagreement (a stand-off) with each other in the following way: the conditionals have identical antecedents and contradictory consequents. Moreover, no third party would have a reason to choose between the two conditionals (deeming one as wrong and the other one as right), because each of the assertions seems to be equally justified (Santos, 2008, Section 1). The phenomenon has been shown to be widespread by a number of authors and so there are numerous examples (Gibbard, 1981; Santos, 2008).[27] For the purposes of the present discussion I will analyse one particular example, given by Bennett.

> Top Gate holds back water in a lake behind a dam; a channel running down from it splits into two distributaries, one (blocked by East Gate) running eastwards and the other (blocked by West Gate) running westwards. The gates are connected as follows: if east lever is down, opening Top Gate will open East Gate so that the water runs eastwards; and if west lever is down, opening Top Gate will open West Gate so that the water will run westwards. On the rare occasions when both levers are down, Top Gate cannot be opened because the machinery cannot move three gates at once. Just after the lever-pulling specialist has stopped work, Wesla knows that west lever is down, and thinks "If Top Gate is open, all the water will run westward"; Esther knows that east lever is down, and thinks "If Top Gate is open, all the water will run eastward". (Bennett, 2003, p. 85)

Clearly both Wesla and Esther speak the truth, yet appear to disagree with each other.[28] Moreover, we can imagine there being a third party, who does not know the settings of the levers, but hears what Esther and Wesla say, and has

[27] Bennett (2003, p. 87) argues that the vast majority of acceptable conditionals with a false antecedent are based upon stand-off situations.

[28] Let us make an informal observation. The proposed analysis indexes conditionals by contexts, but it should be noted that on the assumption that in any given situation an individual (speaker) need not justify their assertion by recourse to the same context as other individuals (in general those will differ), the proposed analysis could just as well be indexed by individuals. Consequently, epistemic considerations could be employed in explaining the differences in the justifications for asserting one conditional instead of its *stand-off counterpart*.

good reasons to believe them. This leads the third party to correctly conclude that the antecedent must be false, i.e., that the Top Gate is in fact closed (see also Priest, 2018, p. 5). It will now be shown how the proposed analysis accommodates such situations. What we essentially want is an analysis that evaluates both (6) and (7) as true at the same world, but according to different contexts. This is precisely what **CS**+ has been tailored to do and what has been argued in Section 1 to be an unattainable feat on the analyses in the contested class. An explanation that involves contextual considerations would be desired, and as it will become clear, the proposed account does this naturally.

6. If Top Gate opens, all the water will run westwards.
7. If Top Gate opens, all the water will run eastwards.

Both Wesla and Esther speak the truth, albeit relative to their contexts, so (6) and (7) can be formulated, with the enthymematic/nominal contextual content revealed, as follows:

6.1. In context *w*: If Top Gate opens, all the water will run westwards.
7.1. In context *e*: If Top Gate opens, all the water will run eastwards.

We can explain how the third party infers that Top Gate is in fact (actually) closed, upon hearing (6) and (7) analysed as (6.1) and (7.1) respectively.[29] What follows is an informal argument, which will be subsequently followed by providing a **CS**+ model that reflects it closely. The third party knows that both conditionals are true, relative to their context. So there is contextual information carried by similarity assignments $r(@, w)$ and $r(@, e)$ corresponding to contexts w and e, respectively ($@$ denotes the actual world), such that in any (closest) world according to $r(@, w)$ where Top Gate is open, all the water flows West, and in any (closest) world according to $r(@, e)$ in which Top Gate is open, all the water will flow East. The third party gathers from what Wesla and Esther say that both lower gates must be open, and from that alone it follows that Top Gate must be closed. If Top Gate were actually open, all the water would flow West and all the water would flow East, which is impossible.[30]

To explicitly demonstrate how the formal model theory fares in giving an account of this scenario, we construct a **CS**+ model. Let us formalize (6.1) and (7.1) by recourse to the set of formulae $For^{\mathcal{C}}$ of the extended language: denote (6.1) with $T >_w W$, (7.1) with $T >_e E$. Let $\mathfrak{A} = (W, \mathcal{C}, r, V)$ be as follows: $W = \{@, j, k\}$,

[29] The reasoning couched on the proposed semantics parallels one given by Priest (2018, pp. 4–5). Whereas Priest appeals directly to information importation in the explanation of contextual disparities, I appeal to similarity assignments that are interpreted as carriers of contextual information.

[30] This explanation mirrors one given by Priest (2018, p. 5) given in terms of information importation.

$\mathcal{C} = \{w, e\}$. Below, in the characterization of $\lesssim_{@,w}$ and $\lesssim_{@,e}$ the ellipses indicate the reflexive cases.

$r(@, w) = (\{@, j, k\}, \{(@, j), (@, k), (j, k), \ldots\})$	Wesla's context.
$r(@, e) = (\{@, j, k\}, \{(@, k), (@, j), (k, j), \ldots\})$	Esther's context.

$V_@(W) = V_@(E) = 1$, $V_@(T) = 0$	Top Gate is closed, since the other two are open.
$V_j(T) = V_j(W) = 1$	Top Gate and West Gate are open.
$V_k(T) = V_k(E) = 1$	Top gate and East Gate are open.

It is easy to check that both $T >_w W$ and $T >_e E$ are true at the actual world, as required. Not only is the analysis adequate, but it also avoids the pitfall of committing to $T >_c (W \wedge E)$ at the actual world for any context c, which is certainly desirable. The argument for this runs along the same lines as the counterexample to *Adjunction of Consequents* given in the previous section.

4. Conclusion

It should be clear that the proposed account in this article merely extends the traditional accounts, e.g., the Stalnaker-Lewis analyses of conditionals. That is, it merely provides an extension of the traditional accounts in a manner that amends their context related difficulties. However, there is no fundamental tension between what is proposed here and the traditional accounts, other than accounting for the contextual differences that these accounts fail to accommodate in their respective formal semantics.[31]

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[31] As a vindication of this we could take what Stalnaker (2017) himself stated in a seminar in Milan. He affirmed the suggestion of an audience member that his solution to the difficulties raised by Gibbardian Stand-Offs would be by allowing the world selection function to differ from speaker to speaker. This is essentially what I propose, but in a more general fashion.

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REFERENCES

Bennett, J. (2003). *A Philosophical Guide to Conditionals*. Oxford: Oxford University Press.

Berto, F. (2017). Impossible Worlds and the Logic of Imagination. *Erkenntnis*, *82*(6), 1277–1297.

Berto, F. (2014). On Conceiving the Inconsistent. *Proceedings of the Aristotelian Society*, *114*(1pt1), 103–121.

Blackburn, P., Rijke, M., Venema, Y. (2001). *Modal Logic* (Cambridge Tracts in Theoretical Computer Science). Cambridge: Cambridge University Press.

Chellas, B. (1975). Basic Conditional Logic. *Journal of Philosophical Logic*, *4*, 133–228.

Gabbay, D. M. (1972). A General Theory of the Conditional in Terms of a Ternary Operator. *Theoria*, *38*(3), 97–104.

Gibbard, A. (1981). Two Recent Theories of Conditionals. In W. L. Harper, R. Stalnaker, G. Pearce (Eds.), *Ifs* (pp. 211–247). Dordrecht: D. Reidel Publishing Co.

Goodman, N. (1983). *Fact, Fiction, and Forecast* (Fourth Edition). Cambridge, MA: Harvard University Press.

Lewis, D. (1971). Completeness and Decidability of Three Logics of Counterfactual Conditionals. *Theoria*, *37*, 74–85.

Lewis, D. (1973) *Counterfactuals*, Oxford: Blackwell.

Lewis, D. (1981) Ordering Semantics and Premise Semantics for Counterfactuals. *Journal of Philosophical Logic*, *10*(2), 217–234.

Lewis, David (1986). On the Plurality of Worlds. Hoboken, NJ: Wiley-Blackwell.

Mares, E. D. (1997). Who's Afraid of Impossible Worlds? *Notre Dame Journal of Formal Logic*, *38*(4), 516–526.

Mares, Edwin D. (2004). Relevant Logic—a Philosophical Interpretation. Cambridge: Cambridge University Press.

Montague, R. (1970). Pragmatism. In R. Klibansky (Ed.), *Contemporary Philosophy I: Logic and Foundations of Mathematics* (pp. 102–122). Florence: La Nuova Italia.

Nolan, D. (1997): Impossible Worlds: A Modest Approach. *Notre Dame Journal of Formal Logic*, *38*(4), 535–572.

Nute, D. (1980). Topics in Conditional Logic. Dordrecht: D. Reidel Publishing Co.

Popieluch, M. (2019). *Context-Indexed Counterfactuals and Non-Vacuous Counterpossibles.* (Unpublished doctoral dissertation). The University of Queensland, Australia.

Priest G. (2008) *From If to Is: An Introduction to Non-Classical Logic* (Second Edition). Cambridge: Cambridge University Press.

Priest, G. (2018). Some New Thoughts on Conditionals. *Topoi*, *27*, 369–377.

Quine, W. (1960). *Word and Object*. Cambridge, MA: MIT Press.

Santos, P. (2008). Context-Sensitivity and (Indicative) Conditionals. *Disputatio*, *2*(24), 295–315.

Scott, D. (1970). Advice on Modal Logic. In K. Lambert (Ed.), *Philosophical Problems in Logic* (pp. 143–173), Dordrecht: D. Reidel Publishing Co.

Stalnaker, R. (1998). On the Representation of Context. *Journal of Logic, Language, and Information*, *7*(1), 3–19.

Stalnaker, R. (1999). Context and Content: Essays on Intentionality in Speech and Thought. Oxford: Oxford University Press.

Stalnaker, R. (1968). A Theory of Conditionals. In N. Rescher (Ed.), *Studies in Logical Theory (American Philosophical Quarterly Monographs 2)* (pp. 98–112). Oxford: Blackwell.

Stalnaker, R. (2017). *Counterfactuals and Practical Reason* [Conference presentation]. Casalegno Lectures (2017). Retrieved from: https://youtu.be/-PKCSZ_yY7A

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Article

FELIPE MORALES CARBONELL *

TOWARDS SUBJECT MATTERS FOR COUNTERPOSSIBLES[1]

SUMMARY: In this paper, I raise the problem of dealing with counterpossible conditionals for theories of subject matter. I argue that existing accounts of subject matter need to be revised and extended to be able to a) provide reasonable (potentially non-degenerate) verdicts about what counterpossibles are about, b) explain the intuition that counterpossibles are in some sense about what would happen if the antecedent were true, and c) explain in what sense counterpossibles can be about individuals. I sketch how one could extend atom-based and way-based theories of subject matters to handle the problem. Then, I raise the problem that it might be desirable for a theory of subject matter to prevent the inference that certain counterpossibles are about the kinds of things that they seem to mention.

KEYWORDS: counterpossibles, conditionals, subject matter, topic-transparency, subject-predicate subject matters, atom-based subject matters, way-based subject matters.

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1. Introduction

Counterpossibles have caused a lot of headache for philosophers. Mostly, though, the concern with them has been whether they can be true non-vacuously, against the standard or orthodox view.[2] There is, however, a different worry, that sometimes is hinted at in discussions of counterpossibles, but that to my knowledge has never been treated with the full explicitness that it deserves. It is this: what are counterpossibles about?

The question is worrisome because on the absence of an answer, room is left open for debates on counterpossibles to devolve into into discussions about who changed the subject, and how. The problem is already there when we have to evaluate impossible statements in general. Gendler observes that when faced with statements that purport to describe impossibilities, the principle of charity might force us to think that people who utter them must have changed the subject:

> If someone comes up to me and says "Twelve both is and is not the sum of five and seven," it seems that I have no choice but to reinterpret one or more of her terms. Whatever she is talking about, she cannot mean by "twelve" and "both" and "is" and "and" and "not" and "sum" and "five" and "seven" what we mean by those terms. It just does not make sense to say that twelve both is and is not the sum of five and seven; and since I cannot make sense of what it would be for twelve both to be and not to be the sum of five and seven, I surely cannot imagine a story in which it is true that twelve both is and is not the sum of five and seven. (Gendler, 2000, p. 67)

The challenge is typical (cf. Williamson, 2007, p. 177). People who defend the use of counterpossible reasoning need to have an answer to the objection that they are changing the subject. Without a theory of subject matters for counterpossibles, any answer can be criticized as ad hoc or as guided by unreliable intuitions. Note that a similar point can be made against those who object to counterpossible reasoning on the grounds of the change of subject objection, since they also lack such theory. While we can be guided by intuitions in the construction of a theory, we cannot be satisfied with them.[3]

What sort of pre-theoretical intuitions do we have about the subject matter of counterpossibles? Take a counterpossible conditional like:

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[2] Stalnaker (1968; 1987), Lewis (1973), Williamson (2007; 2018), Emery and Hill (2016), and Vetter (2016) defend the orthodoxy. Nolan (1997), Vander Laan (2004), Kim and Maslen (2006), Yagisawa (2010), Brogaard and Salerno (2013), Kment (2006; 2014; 2016), Jago (2014), Bernstein (2016), Priest (2016), Berto et al. (2018), Kocurek (2018), Weiss (2019), Locke (2019), Tan (2019), Berto and Jago (2019), and Kocurek and Jerzak (2021) defend heterodoxy. Cf. also Baker (2007).

[3] Dialectically, however, the burden seems to be on the defender of counterpossible reasoning: the objector could be satisfied with a much weaker theory that assigned degenerate subject matters to all counterpossibles, while the defender may need to show that different counterpossibles have different subject matters.

1) If 1 + 1 were 3, 1 + 2 would be 4.

A naive answer to the question "what is it about?" is readily at hand: this counterpossible is about the numbers 1, 2, 3 and 4, because those are the numbers that it mentions (this is the Mention-Criterion, or MC for short; cf. Ryle, 1933). The same answer can be given for other counterpossibles:

2) If the laws of logic were different, different argument forms would be valid (where we can say that the counterpossible is about the laws of logic and argument forms),
3) If I had different parents, I could have been born in Marseilles (about me, about my parents, about Marseilles).

The naive answer, however enticing it may be, cannot be the full answer. Suppose that these counterpossibles are about these things (and this is an assumption that we may have to drop, as I will argue later). Intuitively, it seems to me correct that in general these counterpossibles are *also* about what would happen, were the antecedent true.[4] While this strikes as something that is generally true about the subject matter of counterfactuals (call it the Counterfactual Subject Matter Principle, or CSP for short), we cannot capture it in the same way as the MC. How to do it?

This paper is largely exploratory in character. It is structured as follows. In Section 2, I will lay down some desiderata for a theory of subject matter of counterpossibles, and examine how current theories of aboutness (subject-predicate-based, atom-based and way-based) can deal with the problem of the subject matter of counterpossibles. To anticipate my assessment: these theories are inadequate to meet those desiderata. Naturally, it is worth asking whether suitable modifications to these theories could make them fit for purpose. In Section 3, I will examine how we can enrich atom-based accounts of subject matters with structure, and how this kind of solution can fare with the issue of the subject matter of counterpossibles. In Section 4 I will sketch a ways-based theory of subject matters where the subject matters of counterpossibles are patterns of counterfactual variance in enriched modal spaces. Both atom-based and way-based theories are shown as viable candidates for a theory of subject matters for counterpossibles, partially vindicating the position of defenders of counterpossible talk. However, in Section 5 I will suggest that a theory of subject matters for counterpossibles should allow for counterpossibles to fail to be about the items that they mention. To make that work, we should favor either a way-based approach or some form of pluralism about subject matter. But it would also undermine some of the intuitions of defenders of counterpossible talk.

[4] Let me fix a bit of typographical convention: I will use sans serif labels to denote descriptive names for subject matters.

2. The State of Things

In what follows, I will assume that a theory of subject matter for counterpossibles can be developed as an extension of a theory of sentential subject matter.[5] There are various alternative accounts of subject matters to draw on. With Hawke (2018), who provides a nice overview of the theories of subject matter available, I will distinguish between subject-predicate-based, atom-based, and way-based conceptions of subject matters.[6] Hawke evaluates the different theories in terms of a series of desiderata. Two of those are particularly relevant here: first, most necessary statements are about something, but not about everything, and second, most impossible statements are about something, but not about everything.[7]

For our purposes here we need to introduce three more desiderata, in line with our brief discussion in the introduction. A theory of subject matters for counterpossibles should, in my estimation:

a) For any counterpossible (and more generally, for any counterfactual), provide with a reasonable (definite, and potentially non-degenerate) verdict about what its subject matter is.[8]

b) Capture the intuition behind the CSP, so that for any counterpossible/counterfactual $\varphi \mathrel{\Box\!\!\rightarrow} \psi$, its subject matter includes what would be true, were φ true.[9]

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[5] The assumption may not be innocent. Edgington (2008), for example, argues that indicative and subjunctive conditionals are not truth-stating and should be treated suppositionally. In the context of a theory of this sort, it may be natural to treat the subject matter of counterpossibles separately from sentential subject matter also. Edgington herself, however, seems to suggest that the subject matter of counterfactuals is the consequent under the supposition of the antecedent. It seems like this can be captured without disentangling the subject matter of counterfactuals and truth-stating sentences. There might be viable theories where this is not the case, but none exists in the literature as far as I can tell.

[6] The distinction has more to do with the metaphysical frameworks they are embedded in than the kind of resources that the approaches make use of to account for subject matters. For example, Hawke himself shows how from some atom-based accounts like his own we can derive both subject-predicate and ways-based theories.

[7] Cf. Hawke's (2018, p. 7). Note that Hawke qualifies with "most", so he leaves it open for some necessities and impossibilities to be either about nothing or about everything. But crucially, he thinks that claims of the forms $\varphi \lor \sim\varphi$, $a = a$, $\varphi \land \sim\varphi$, and $a \neq a$ are about something and not about everything.

[8] Something has a degenerate subject matter when it is about either nothing or anything. The intuition here is that counterfactuals, and by extension counterpossibles, do not (necessarily) have degenerate subject matters.

[9] Arguably, this is an instance of a constraint that Hawke considers but ultimately dismisses in its full generality, which is that we should be able to associate subject matters to questions (Hawke, 2018, p. 7). Not all theorists about subject matter would be willing

c) Capture the intuition behind the MC, so that for any counterpossible/counterfactual, the theory says what individuals it is about, if it is about any.

Clearly, a theory that does not meet a) cannot meet b) or c). As our discussion proceeds, I will consider some further constraints that a theory of subject matters for counterpossibles should respect (in particular, I will return to this point in Section 5), but the above will give us a baseline for evaluation.

In the next subsections I will consider how subject-predicate (2.1), atom-based (2.2) and way-based (2.3) accounts can deal with our desiderata.[10]

2.1 Subject-Predicate Conceptions

The basic idea of the subject-predicate approach is that subject matters of sentences are the sets of the objects that serve as the subjects of predication in those sentences. For example, the subject matter of "John loves Katy" is {John, Katy}, since in it John is a subject for the predicate "loves Katy" and Katy is a subject for the predicate "is loved by John". A more concrete version of the view can be found in (Perry, 1986). The main ideas there are that: 1) propositions are sets of situations that verify an issue (along the lines of Barwise and Perry's [1983] situation theory), and 2) what a proposition is about is the set of the objects that constitute every member of the proposition. This rationalizes the result above: John and Katy are constituents of every situation that verifies "John loves Katy", so the subject matter of "John loves Katy" is {John, Katy}.

Problems immediately arise when it comes to disjunctions such as "John is in love or Katy is happy", which is verified by situations where John is in love but Katy is not happy, situations where Katy is happy but John is not in love, and situations where John is in love and Katy is happy. There is nothing that is a constituent of all the situations that verify the disjunction, so the subject matter of the disjunction is the empty set (∅). We can interpret this result in two ways: either the disjunction is about everything, given that ∅ is a member of all sets, or it has no subject matter. The issue also affects material conditionals, since they are equivalent to disjunctions.

Similar troubles will beset counterfactual conditionals, and thus counterpossibles. On the one hand, if what a sentence is about is a set of objects, we should expect a material conditional and a counterfactual to be about the same things (call this the Conditional Likeness Principle, or CLP). Intuitively, "If John

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to take this as a necessary constraint on a theory for the subject matter of counterpossibles (for example, Hawke would not be too worried if we cannot associate some subject matter to some question—for them there is no reasonable expectation that there would be). However, I think that the connection between subject matters and questions should be central to our understanding of both, so I will take it as a strike against a theory of subject matters that is not able to say something about the point.

[10] I will only give summary sketches of the accounts. An interested reader should refer to Hawke's (2018) for a more formal overview.

is not in love, then Katy is happy" and "If John were not in love, Katy would be happy" involve the same individuals. But since material conditionals have the problem we just saw, if we assume the CLP, the counterfactual would have ∅ as its subject matter, since the material conditional has it as its subject matter.

There are various ways to solve the issue. The first is to modify the notion of subject matter at play in the theory so that it gives non-degenerate results in the disjunctive case. Barwise (1989, p. 66) seems to go for this option when he says that the subject matter of a proposition is "anything that is a constituent of one of the *possible* facts used to characterize the situation" (emphasis mine). Hence, the subject matter of "John is in love or Katy is Happy" is the set {John, Katy}, as we would expect, and we can apply the CLP to get the desired result for conditionals.[11] The second way to solve the issue is to reject the CLP, to leave space for counterfactual subject matters to diverge from the subject matter of material conditionals or other extensional structures. In this case it is necessary to provide an account of subject matters for counterfactuals that is independent from that of disjunction or the material conditional. The CLP is not a principled way to identify the subject matter of counterfactuals; the reason why the subject matter of the counterfactual includes John and Katy is not that the counterfactual has some relation to some material conditional, but comes from the features of the counterfactual itself. The CLP is a constraint on what kinds of subject matters conditionals could have in a language that contains both material and counterfactual conditionals. Applying the CLP to identify the subject matter of a counterfactual requires that we assume that this constraint is met. This cannot be the case if we have an issue with the subject matter of even material conditionals. So a proper solution to the issue will have to specify the subject matter of disjunctions in a more reasonable way, and provide an independent account of the subject matter of counterfactuals. If that can be done, there may not be a need to reject the CLP or a similar principle (for example, it may be possible to say that for any counterfactual $\varphi \mathrel{\Box\!\!\rightarrow} \psi$ with subject matter m there is a material conditional $\varphi \rightarrow \psi$ such that its subject matter $m' \subset m$).

Before we try to come up with a construction that can handle counterfactuals, we should pay attention to the issues that would appear once we try to meet our second desideratum. There are reasons to think that a subject-predicate account cannot meet it even in principle. In fact, there is reason to think that it cannot distinguish between the subject matter of even simpler sentences (for example, "John loves Ann" and "Ann hates John" end up having the same subject matter). In the case of counterfactuals, by the CSP we would have that the "if Samantha were to ask Kira, she would know the answer" has as part of its subject matter what would happen, were Samantha to ask Kira, and "if Samantha were to shoot Kira, she would go to jail" has as part of its subject matter what would happen, were Samantha to shoot Kira. But there does not seem to be any way to say that

[11] In fact, as we will see, this move gives a theory that is closer to the atom-based conception.

these counterfactuals are about different things in any implementation of the subject-predicate approach. A different way to put this is that what we want is a way to say that counterfactuals talk about what would happen, not merely to whom it would happen. [12] This simply is a consequence of the inter-independence of our second and third desiderata.[13] Since the approach seems to be a non-starter for counterfactual conditionals in general, it cannot even begin to be a satisfactory account of the subject matter of counterpossibles.[14]

2.2 Atom-Based Conceptions

In atom-based conceptions of subject matters, the subject matter of complexes is the combination (either the set-union or some kind of fusion) of the subject matters of the atoms that constitute those complexes. Thus, for example, the subject matter of a sentence of the form $p \wedge q \vee \sim q$ is the union of the subject matters of p and q. This neatly solves the issue with disjunctions that affects the subject-predicate view (as I pointed out already, Barwise's version of the subject-predicate view, which solves it, simply adopts union instead of intersection for conjunctions and disjunctions), and offers a way forward for handling conditionals of at least some sorts.

We still need a way to obtain the subject matter of counterfactuals. The astute reader will note at once that the atom-based view does not necessarily solve the issue of meeting our second criterion, for the simple reason that it might also assign mere sets of individuals to atoms. In what follows I will consider how atom-based views that do not shoot themselves in the foot in the obvious way fare with counterfactuals and counterpossibles. In particular, I will examine how theories along the lines of Fine's (2020) state-based theory of subject matters and Hawke's (2018) issue theory of subject matters can handle the problem.

Fine's (2020) proposal makes use of the notion of states, which are not complete worlds, but "situations"; he treats worlds as a special case (worlds are consistent and complete, whereas states need not be either). In Fine's theory, not all states need to be possible, and states that necessarily co-obtain need not be identical. States can be constructed unrestrictedly by fusion: for any states $|A|$ and $|B|$ in logical space, there is a state $|A \sqcup B|$ (notationally, I will use $\|A|,|B\|$ in what

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[12] Both Perry (1986) and Barwise (1989, Chapter 5) suggest that we should think of conditionals differently from non-conditional statements. In their view, rather than being about situations and their constituents, conditionals are about the relations between types of situation, or what they call *constraints*.

[13] This can illuminate what a principle like the CLP actually is about. The CLP can only be taken as a principle about *the things* that conditionals talk about, not about what the conditionals are about simpliciter. So it cannot be used to fully identify the subject matter of counterfactuals.

[14] As a reviewer notices, the point can generalize into a more general argument against subject/predicate conceptions of subject matter.

follows).[15] Fine simply identifies subject matters with states: the subject matter of φ is the fusion of the exact verifiers and exact falsifiers of φ.

Because the theory characterizes subject matters in terms of verifiers and falsifiers, it is forced to appeal to the semantics of the relevant expressions. In the case of the subject matter of counterfactuals and counterpossibles more specifically, the theory depends on the semantics for counterfactuals being able to make distinctions between counterfactuals with different impossible antecedents. This means that the semantics for counterfactuals cannot be the standard one. Fine (2012) offers a divergent semantics for counterfactuals that makes use of states instead of worlds, and in his (2021) sketches a way to modify this proposal in order to handle counterpossibles. Fine himself has not provided a theory of subject matters for counterpossibles, so I will now attempt to project one from some of the materials that he has made available. What follows is an admittedly simplified version of what Fine's theory could be, but it will serve to illustrate some problems that a theory of this kind could face.

The core idea of Fine's semantics for counterfactuals is that counterfactuals are primarily concerned with the "outcomes" of states. The antecedents of counterfactuals expresses conditions of change (themselves states) for states, and the consequent expresses something about the outcome of those changes (also states). As a rough characterization of the truth conditions of counterfactuals, Fine claims that a counterfactual $\varphi \mathrel{\Box\!\!\rightarrow} \psi$ is true iff any possible outcome of an φ-state contains a ψ-state.

More explicitly, Fine's (2012, p. 237) view is that $\varphi \mathrel{\Box\!\!\rightarrow} \psi$ is true at a state w iff u inexactly verifies (is partially relevant to) ψ whenever t exactly verifies (is wholly relevant to) φ and u is a possible outcome of t relative to w. When in the truth condition the φ-states are restricted to possible states, the semantics gives the same verdict as the orthodox semantics when it comes to counterpossibles. To account for non-vacuous counterpossibles, Fine (2021) suggests the following. States have mereological structure, so they can be decomposed into other states. To calculate the outcomes of impossible states we look into the outcomes of states that they might decompose into—this will mean that impossible states for which we can calculate the outcome normally will be decomposed into possible states for which we can calculate the outcome (*ex hypothesi*).[16] For example, with an atomic p, the impossible state $|p \,\&\, {\sim}p|$ decomposes into $|p|$ and $|{\sim}p|$; that is, $||p|, |{\sim}p||$. For an impossible state s that decomposes into possible states $s_1, s_2, \ldots, s_n$ there will be states $t_1^1, t_2^1, \ldots, t_n^1$ which are the possible outcomes of each of $s_1, s_2, \ldots, s_n$. Fine's proposal is to take the fusion of those possible outcomes as the outcome of s. This outcome may be an impossible state. Consider the counterfactual:

[15] For an elaboration of the construction of this state space, see (Fine, 2021).

[16] In principle, there could be decompositions of impossible states into other impossible states for which we can calculate the outcome, but it is not clear from Fine's proposal how this could be made sense of.

4) If I were vegan and not-vegan, I would both eat exclusively only non-animal based foods and non-exclusively animal based foods.

The antecedent is verified by the impossible state ||am vegan|, |am not vegan||, which decomposes into the possible states |am vegan| and |am not vegan|. |eat exclusive non-meat| is a possible outcome of |am vegan|, and |eat non-exclusive meat| is a possible outcome of |am not vegan| (for simplicity, let us suppose that these are the only possible outcomes). Consequently, the outcome of ||am vegan|, |am not vegan|| is the (impossible) state ||eat exclusive non-meat|, |eat non-exclusive meat||. All the possible outcomes of the φ-state contain a ψ-state, so the counterfactual comes out as true.

Perhaps more interestingly, in this view some counterpossibles come out as false. For example:

5) If Hobbes had found a counterexample to Fermat's Last Theorem, he would have squared the circle.

Roughly, the idea here is that |squared circle| is not the possible outcome of any plausible decomposition of |counterexample found|, so the counterfactual comes out as false.

We might reason that in that case we do not actually need to decompose |counterexample found|: since |squared circle| is itself impossible, it cannot be the possible outcome of any possible state). However, this creates a complication with

6) If Hobbes had found a counterexample to Fermat's Last Theorem, he would have found a counterexample to Fermat's Last Theorem.

Since |counterexample found| is impossible, by parity of reasoning it cannot be the outcome of any possible state, so it is not the outcome of any decomposition of |counterexample found|, and the counterfactual seems to evaluate as false. However, intuitively, the counterfactual should come out as true (even Berto, French, Priest, Ripley, 2018 accept reflexivity). To make this work Fine has to add explicitly the assumption that impossible states always decompose into possible states, or that it does given the conditions of the case (however those are spelled out). Then, the possible states into which |counterexample found| decomposes must be possible outcomes of the possible states into which |counterexample found| decomposes, just by reflexivity on possible states, which is uncontroversial. The reason why (5) is false is that the possible states into which |squared circle| decomposes are not possible outcomes of the possible states into which |counterexample found| decomposes. Since the falsity of the counterfactual depends on the decompositions of its components, other false counterpossibles will have different falsitymakers.

If we plug this machinery into the Finean conception of subject matters we get that the subject matter of the counterfactual is exactly the fusion of its verifiers and falsifiers. Since different counterpossibles can be made true and false in different ways, we can distinguish between the subject matter of different counterpossibles.

While the full Finean picture (the union of his theory of subject matters and his semantics for counterfactuals) can give an account of the subject matter of counterpossibles, one may wonder if the theoretical cost of adopting Fine's framework is too high.[17] Besides, the theoretical costs of the framework are difficult to assess because some of its central notions are under-specified; for example, the notion of an outcome is not fully delineated, so it it not clear how fit for generalizations it is. Consider, for example, mathematical counterpossibles: what is the "outcome" of a mathematical antecedent? What we need is something that can support the structure of the theory across domains—and that, whatever it is, may not match with Fine's notion of an outcome.[18] Furthermore, there is no obvious way to understand the connection between the kind of states that the theory predicts as the subject matters of counterpossibles, and the kind of question that we want to capture by our second desideratum.[19]

Another problem is that the assumptions about the decomposition of states that Fine's semantics of counterfactuals requires are highly controversial. Remember the attempted solution to the problem of validating reflexivity, which required the assumption that impossible states can always be decomposed into possible states. This is unsatisfying, because: 1) there is no principled way to decompose impossible states like |counterexample found|, and 2) the assumption may seem ad hoc (why could not there be "primitive" impossible states, and why, for example, could not |counterexample found| be a primitive impossible state?).[20]

A different problem is that it is not clear that the theory even yields the correct predictions about truth values for counterpossibles: since states are easy to come by by fusion, the theory might over-generate candidate outcomes, which in

[17] Along similar lines, Yablo (2018, p. 1497) raises the worry that the benefits of moving from a framework of worlds (which are relatively well understood) to the framework of states might not be worth it. Fine (2020), of course, argues otherwise. Here, I will not attempt to adjudicate what approach will fare better in terms of the cost/benefit analysis of theoretical virtue.

[18] Fine (2012, p. 237) warns that we should not be misled by the term "outcome", and that in some cases (such as the case of "if his peg had been round then it would not have fit the hole") the outcome-relation "could be taken to be more logical or conceptual in character", so his notion is more general than one could be led to think from his treatment of causal examples, and consequently we should not treat the worry raised here as a knockdown argument against the approach.

[19] Relatedly, Hawke (2018, p. 25) raises the question why we should think of Finean states as subject matters at all.

[20] Fine (2021, pp. 154–156) is aware of the point, and sketches some ways to deal with such "modal monsters", although it is not clear to me how they would solve the current problem concretely (of course this does not mean there is no way).

turn gives more ways for counterpossibles to come out as true.[21] In fact, the application of unrestricted fusion to construct impossible states should be regarded with suspicion. For the kind of impossible states that we need to consider, the application of fusion is the only reason to think that they may exist; we cannot say that, given that those impossible states exist, they must have been built by fusion from other states. The introduction of those states to the ontology is accepted merely on the basis of theoretical benefits.

Hawke (2018) proposes a different atom-based theory of subject matters, the issue theory. The guiding idea behind this proposal is that subject matters are systems of distinctions, which are associated to issues concerning whether the world is a certain way or not. The same issue can be answered in different ways by different worlds: those are the ways things are in those world relative to the pertinent subject matter. This idea gives itself more naturally to atomic sentences, which locate a world as distinguished concerning a certain subject matter (in the ordinary sense). More generally, then, subject matters of sentences will be sets of distinctions. For complex sentences, these sets are built by union of the subject matters of the atoms that compose them.

The issue theory of subject matters seems like a solid contender against Fine's. Hawke shows that both theories meet equally well a series of desiderata for theories of subject matters. However, an advantage of Hawke's account is that it does not require the heavyweight state ontology that the Finean proposal requires, and the primitive notions at play are reasonably well specified (which was not the case with the Finean notion of an outcome, as we saw above). The theory is also able to give verdicts about subject matters without having to appeal to the semantic and meta-semantic properties of the sentences at hand. But how well does it fare with counterpossibles?

In the issue-theorist's proposal, distinctions are modeled as ordered tuples of general and individual concepts. Thus, the topic of $Fa \wedge Fb$, $t(Fa \wedge Fb)$, is $\{\langle\mathfrak{F}, \mathfrak{a}\rangle\} \cup \{\langle\mathfrak{F}, \mathfrak{b}\rangle\}$, that is, $\{\langle\mathfrak{F}, \mathfrak{a}\rangle\}$, $\{\langle\mathfrak{F}, \mathfrak{b}\rangle\}$. The approach allows the subject matter of different impossibilities to be distinguished: for example, $t(Fa \wedge {\sim}Fa)$ is not the same as $t(Gb \wedge {\sim}Gb)$. With this, one can also distinguish between conditionals with different impossible antecedents. What one cannot do is to distinguish between indicative and subjunctive conditionals.[22] Without this ability, the approach cannot meet out second desideratum, assuming that indicatives and counterfactuals answer different questions (that is, if "what happens, if *X*?" is different from "what would happen, if *X*?", which strikes me as plausi-

[21] The issue compounds on the problem that it is not entirely clear how we should calibrate the verdicts of a theory of truth conditions for counterpossibles (this is, after all, why the debate on their truth conditions remains). Cf. Williamson's (2021) for some related worries.

[22] Perhaps this needs not concern theorists who argue for a unified treatment of both kinds of conditionals, like Starr (2014).

ble).[23] In any case, even if having a way to distinguish between the subject matter of different classes of conditionals was not sufficient to satisfy the second desideratum, it would be desirable for a theory of subject matters to have this capacity.

2.3 Way-Based Conceptions

In way-based conceptions of subject matters, the notion of a way is taken as a primitive. Subject matters are sets of ways that things could be. The most prominent versions of the approach are Lewis' (1998b, 1998a) and Yablo's (2014).

Lewis' view builds on the idea that worlds that are exactly alike make true the same things with respect to the same subject matters. In worlds where a class of objects *O* is alike, truths about *O* will be alike. Worlds which are exactly alike with respect to a subject matter will make true the same things about that subject matter. Consequently, we can group worlds in equivalence classes that exhaust the ways that worlds can be with respect to those subject matters (they partition modal space in the ways in which worlds can be with respect to them). Lewis proposal is then to identify subject matters with equivalence relations or the partitions of those equivalence classes. Yablo (2014, pp. 27–28) offers two additional characterizations of Lewisian subject matters: i) as specifications of what goes on each world with respect to the subject matter, and ii) as sets of propositions that correspond to questions (the subject matter of "Francis won the championship" is the answer-set to the question "who won the championship?", that is, who won the championship).

How to apply this to counterfactuals and counterpossibles? Consider the following counterfactual:

7) If it had rained yesterday, the plants would not have withered.

Intuitively, as per the CSP, it has as its subject matter what would have happened, had it rained yesterday. In Lewis' account, what that is depends on the account we have for the truth conditions of counterfactuals, since we require those to determine what worlds are exactly alike in what respect to the subject matter: the subject matter of (7) is the partition of ways in which (7) would have been true (or false). Paired with Lewis' own semantics, we have that the subject matter of (7) is the partition of ways in which either (i) in no world it rained yesterday (where the counterfactual would be vacuously true), (ii) in which there is a sphere of worlds *S* such that it rained in some world *s* in *S*, and where it is

[23] Hawke does not consider a language with an intensional conditional, but Berto (2018) does (he considers a language with a strict conditional). There, we are told that the subject matter of a sentence is the set union of the subject matters of the sentence's atoms, so the subject matter of strict conditionals should be equivalent to the set union of the subject matter of the antecedent and consequent (note that this is extrapolation, the explicit theory of subject matters given there does not assign subject matters to strict conditionals explicitly).

true that for all worlds in *S*, if it rained, the plants did not wither (where the counterfactual would be non-vacuously true), or (iii) in which there is a sphere or worlds *S* such that it rained in some world *s* in *S*, and where it is false that for all worlds in *S*, if it rained, the plants did not wither (where the counterfactual would be false). In the Lewisian picture, we evaluate the truth of a counterfactual in the context of a system of spheres of worlds. This system of spheres, which is a partition of logical space, naturally corresponds to a subject matter.

When we apply this to counterpossibles, which are vacuously true, we get the result that the resulting partition contains as its sole member the whole set of possible worlds (in every world, the counterfactual is made true vacuously). Further, all counterpossibles have the same subject matters. Lewis admits the same with regards to contradictions:

> The proposition expressed by a contradiction is about any subject matter because, since there is no way at all for two worlds to give it different truth values, *a fortiori* there is no way for two worlds to give it different truth values without differing with respect to the subject matter. (1998a, p. 121)

But intuitively, we want to say that the subject matters of

8) If I had different parents, I would have lived in Ontario,

9) If Nero had not been Nero, he would have been a butterfly

are different, so Lewis' theory of subject matters plus his semantics for counterfactuals cannot provide an account for the subject matter of counterpossibles.[24] A potential approach to solve this issue is to extend the theory to make use of "impossible" worlds (our previous discussion of Fine's atom-based theory can provide some hints about how this could go). I will return to this later.

Yablo (2014, p. 27) maintains what he takes to be the central idea of Lewis account, that a subject-matter is "a system of differences, a pattern of cross-world variation". His proposal extends Lewis' theory by switching equivalence classes by similarity classes, so that instead of partitions of logical space we get divisions of logical space, which can overlap.[25] Alternatively, he elaborates a notion of sentential subject matters that makes use of the notion of truthmaking: a sentence *s*'s *subject matter* $\vec{s}$ is the set of its potential truthmakers, a sentence *s*'s *subject anti-matter* $\overleftarrow{s}$ is the set of its potential falsitymakers, and its *overall subject matter* is the unordered-pair $\{\vec{s}, \overleftarrow{s}\}$.

While this is an improvement over Lewis' theory, there are problems with this as well. The most troublesome for the problem of counterpossibles is that

[24] Hawke (2018, p. 709) raises the same problem for way-based theories with the case of logical validities: the subject matter of any expression of the form $\varphi \vee \sim\varphi$ is everything.

[25] In a footnote he advances that we should be more liberal still, replacing divisions with what he calls covers.

Yablo's account may still be inadequate to handle some hyperintensional contexts.[26] Fine raises the point as a reason to favor his state-based account:

> If one thinks of a subject matter as being given by a set of states, then this means that certain hyperintensional differences in subject matter may be lost once one moves to the corresponding equivalence or similarity relation on worlds. One subject matter may be *mathematical truth*, another *metaphysical truth*, each constituted by certain necessary states. The subject matters are quite different and yet the corresponding relations will be the same, since all worlds will agree on the mathematical facts and all will agree on the metaphysical facts. (2020, p. 151)

If the subject matter of counterfactuals is the division of ways in which the counterfactual can be similarly true or false, we will again have the problem that counterpossibles with a standard semantics will share their subject matter: the overall way the world is.[27] I will return to way-based accounts in Section 4.

3. Enriching Atom-Based Accounts With Structure

Summarizing the previous section: some straightforward ways to deal with the issue of the subject matter of counterpossibles utilizing existing approaches are deficient. On the one hand, neither subject-predicate nor atom-based approaches seem able to meet our second desideratum. On the other hand, existing way-based approaches seemingly cannot provide with a way for different counterpossibles to be about different subject matters, and thus fail to meet the first desideratum. We need something else.[28]

In this section, I will sketch one way that we could proceed to extend the atom-based approach so that it can handle the second desideratum. In the next section, I will sketch how we could proceed starting from a way-based approach.

The problem we are facing is really that the theories of aboutness we have available are theories of the subject matter of a limited range of sentences, namely those that can be constructed from atoms by negation, conjunction and disjunction. Conditionals fit awkwardly in this context (the material conditional can

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[26] Yablo (2014, pp. 92–94) hints at some ways to make sense of impossibilities; they can be understood as relatively possible with respect to a limited set of constraints, even if they are impossible when all constraints are considered. So, roughly, at least part of the subject matter of counterpossibles may be the part of what they say that does not take into account those constraints.

[27] Hawke (2018, p. 15) notes that there are some variations of Yablo's proposal that can circumvent the issue, but these also fail for other reasons.

[28] In more recent and unpublished work, both Fine and Yablo have provided more refined versions of their theories of subject matter (for example, Yablo has explored ways to account for sub-sentential subject matter that could be illuminating for our present discussion; cf. also Yablo's [2020] account of aboutness for sentences involving fictional names). In this paper I do not have enough room to deal with how those adjustments could treat the issue of the subject matter of counterpossibles.

be captured, but some might have qualms about it being a correct rendering of indicative conditionals). Intensional conditionals fit the picture only if we treat them as atoms, or as somehow equivalent to other complex constructions like conjunctions or disjunctions. The general problem is that in intensional conditionals structure matters in a way that cannot be captured by the usual rules of topic composition for propositional operators—where for any operator pair of distinct propositional operators $<\circ_1, \circ_2>$, $t(\varphi \circ_1 \psi) = t(\varphi \circ_2 \psi)$. My guiding intuition, on the contrary, is that the subject matter of $\varphi \;\Box\!\!\rightarrow \psi$ is not *just* the subject matter of $\varphi \wedge \psi$ in the best of cases (where they overlap to some extent).[29]

The problem does not actually appear only in the case of subjunctive conditionals. Take an strict conditional, and consider

10) If the question was open yesterday, it has not been answered today,

11) Necessarily, if the question was open yesterday, it has not been answered today.

The first is compatible with there being a world where the question was open yesterday (that is, at a counterpart time) and the question having been answered today (at a counterpart time), while the second is incompatible with it. For the first it is enough that the world of evaluation verifies the conditional, while for the second it is also a concern (in the ordinary sense) whether the same holds across all accessible worlds. So while the first is about what goes on in a world in particular, the second is also about what goes on in the whole arrangement of worlds in logical space. What this suggests is that in some sense, intensional operators such as $\Box$ are not topic-transparent in the same way that negation is: $t(\Box\varphi) \neq t(\sim\varphi) = t(\varphi)$. But if that is so, then why should negation itself be topic-transparent? And why should conjunctions and disjunctions (and material conditionals) be relatively topic-equivalent?

Because we also may have the intuition that, in some sense of "being about", disjunctions and conjunctions, as well as arbitrary formulas and their negations, are about the same things, it seems like something has to give.[30] Despite appearances, in fact we are not forced to make a choice. The way out of the problem, or at least the way out that I will endorse here, is to say that there are different senses of "being about" and that when those are properly distinguished there is no tension. In fact, theories of aboutness like Hawke's already distinguish between a general sense of aboutness (in the case of the issue theory, *atomic-aboutness*), and more restricted senses like that of being about something in particular or

[29] Cf. Berto and Özgün's (2021) where they argue that in the case of on-topic conditionals, the subject matter of ψ must be included in a subject matter that contextually extends on the subject matter of φ, that is the subject matter of $\varphi \wedge \psi \wedge \rho$, where ρ is additional information that is contextually relevant. The subject matter of on-topic conditionals is, then, $t(\varphi \wedge \psi \wedge \rho)$. But nothing is said about the topic of off-topic conditionals; one possibility is to take it to be the union of $t(\varphi \wedge \psi)$ and $t(\psi)$.

[30] Perry (1986) and Hawke (2018) both take those as requisite constraints for a theory of aboutness.

objectual-aboutness ("Fido is happy" is about whether Fido is happy—{<$\mathfrak{H}$, $\mathfrak{f}$>}— but also about Fido himself). Note that while objectual aboutness can be recovered from atomic-aboutness, the opposite cannot be done. In other words, objectual aboutness is lossy with regards to atomic-aboutness. Likewise, atomic-aboutness is lossy with regards to richer senses of aboutness, like those where we can distinguish between the subject matter of different types of conditionals with identical atoms.

I will now sketch a way to develop this idea more explicitly. Assume a language $\mathcal{L}$ with a set $\mathcal{L}_{CONST}$ of constants ($c_1, \ldots, c_n$), a set $\mathcal{L}_{PRED}$ of non-structural predicates of arbitrary arity ($P^1_1, \ldots, P^n_1, \ldots, P^1_n, \ldots, P^n_n$), and a set $\mathcal{L}_{STRUCT}$ of "structural" elements of arbitrary arity (this would include negation, the logical connectives, modal operators, conditionals, etc.).[31] The syntax of the language will be as usual, allowing binary predicates to be used both in prefix and infix notation.

Semantically, we follow Hawke's account with some (mayor) differences. A model M is a tuple $<W, O, a, s>$. W is a set of worlds. O is a set of objects. a is an assignment function that maps each $c \in \mathcal{L}_{CONST}$ an individual concept $\mathfrak{c}$, each P in $\mathcal{L}_{PRED}$ a general concept $\mathfrak{P}$, and each $\nabla \in \mathcal{L}_{STRUCT}$ a general concept ∇ that plays the corresponding structural role. s is a function that maps sentences in the language to subject matters (I will call these *prime* subject matters), as follows:

- $s(P_n c_n, \ldots, c_m) = <\mathfrak{P}_n, \mathfrak{c}_n, \ldots, \mathfrak{c}_m>$
- $s(\nabla\varphi) = <\nabla, s(\varphi)>$
- $s(\nabla\varphi_n, \ldots, \varphi_m) = <\nabla, <s(\varphi_n), \ldots, s(\varphi_m)>>$

This allows us to distinguish between the subject matter of conditionals, as we wanted. Assuming that $\rightarrow$ and $\Box\!\!\rightarrow$ are in $\mathcal{L}_{STRUCT}$, $s(\varphi \rightarrow \psi) \neq s(\varphi \Box\!\!\rightarrow \psi)$ (since $\langle\rightarrow, \langle s(\varphi), s(\psi)\rangle\rangle \neq \langle\Box\!\!\rightarrow, \langle s(\varphi), s(\psi)\rangle\rangle$). Furthermore, it is easy to check that the theory also allows us to distinguish between counterpossibles; for example, $s((Fa \wedge \sim Fa) \Box\!\!\rightarrow Gb) \neq s((Gb \wedge \sim Gb) \Box\!\!\rightarrow Ga)$.

We can now define a function, A, that for each sentence yields an atomic-subject matter, which is the set tuples of the form $<P_n, c_m, \ldots, c_{m'}>$ that we get traversing the prime subject matter of the sentence recursively. It turns out that these subject matters are exactly those that the issue-theory predicts (so, $A(\varphi) = A(\sim\varphi)$, $A(\varphi \wedge \psi) = A(\varphi \vee \psi)$, and so on). We also define a function $\mathfrak{O}$, that for

[31] I do not include variables and quantifiers because Hawke's theory does not include them either. While I do not have space to evaluate this possibility, as a reviewer notices, the introduction of quantifiers might provide a different kind of account for the subject matter of counterpossibles, treating them (and other kinds of counterfactuals) as quantified structures of some sort. Yablo's (2014, pp. 61–67) treatment of quantifiers and conditionals could suggest something like this. Plebani and Spaolore (2020) provide accounts of subject matter that include a treatment of quantified sentences, and Badura (2020) suggests that this extension should be made to Hawke's models.

each sentence yields its objectual subject matter, which is the union of all sets of individual concepts contained in each tuple in the sentence's atomic subject matter.

With these modifications, we have a theory that allows us to meet our first and third desiderata, and that it seems to go some way towards meeting the second, by being able to distinguish the subject matter of different types of conditionals. But this is not sufficient yet, because we want to have some reason to think that the structures that the theory predicts as the subject matter of sentences has some systematic link to the kinds of questions that we associated to counterfactuals in the second desideratum. That is: how come $\langle \Box\!\!\rightarrow, \langle s(\varphi), s(\psi)\rangle\rangle$ represents, for example, the question what would happen, were φ the case?

Before trying to tackle this issue, there is a different point that we need to address. Prime subject matters as described in the theory match the syntax of sentences very closely. In fact, they are in a 1:1 mapping to syntax trees. But if subject matters match with syntax, why think that subject matter is a different dimension to meaning, as Yablo (2014), Fine (2016) and others suggest? It is part of the presuppositions of the various approaches to subject matter that it represents some stopping point in the continuum between extension and syntax, to the right of where intension would lie, but never at the very extreme.[32] In truth, the theory we have can be seen as the top of a lattice of different theories of subject matters that yield different prime subject matters, where we can find a theory where s is replaced by something that is functionally equivalent to A, and also a theory where s is replaced that something that is functionally equivalent to $\mathfrak{D}$. At different points in the lattice, different syntactic elements are "ignored" so that prime subject matters do not capture them.

To implement this idea, we could add a designated element ○ ("blank") to our models. We could then adjust s so that it behaves differently if some elements have been "blanked out" by the a function:[33]

$$s(P_n c_n, \ldots, c_{m'}) = \begin{cases} \{\mathfrak{P}_n\} & \text{if } \forall c \in c_m \ldots c_{m'}, a(c) = \circ \\ \{\langle \mathfrak{P}_n, \mathfrak{c}_m \ldots \mathfrak{c}_{m'} \rangle\} & \text{if } a(P_n) \neq \circ \\ \{a(c) \mid c \in c_m, \ldots, c_{m'} \text{ and } a(c) \neq \circ\} & \text{if } a(P_n) = \circ \\ \emptyset & \text{otherwise} \end{cases}$$

$$s(\nabla\varphi) = \begin{cases} s(\varphi) & \text{if } a(\nabla) = \circ \\ \{\langle \overline{\nabla}, s(\varphi) \rangle & \text{otherwise} \end{cases}$$

$$s(\nabla\varphi_n, \ldots, \varphi_m) = \begin{cases} \bigcup s(\varphi), \ldots, s(\varphi_m) & \text{if } a(\nabla) = \circ \\ \{\langle \overline{\nabla}, \langle s(\varphi_n), \ldots, s(\varphi_m) \rangle\rangle\} & \text{otherwise} \end{cases}$$

[32] I take this observation from Leitgeb's (2018, p. 4).

[33] Some structures that in the previous version of the theory were simply tuples have been wrapped into sets here. This is done so that in cases where structural elements are blanked out, the output of S is uniform.

The kind of theory of subject matters we will get depends on what syntactic elements of sentences are blanked out by the *a* function. For example, the theory we had sketched before does not blank out anything, while the issue-theory blanks out all members of $\mathcal{L}_{STRUCT}$ (in this case, prime subject matters and atomic subject matters coincide). When $\mathcal{L}_{STRUCT}$ is not empty, there might be many intermediate theories that blank out only some of the elements of $\mathcal{L}_{STRUCT}$, where prime subject matters and atomic subject matters do not coincide.[34]

Figure 1

Some Theories of Subject Matters

	S_1		
	↓		
	$S_{1.1}$		$a(\sim) = \circ$
	↓		
	$S_{1.3}$		$a(\wedge) = a(\vee) = \circ$
	↓		
	$S_{1.4}$		$a(\Box\!\!\rightarrow) = \circ$
	↓		
	S_2		$a(\mathcal{L}_{STRUCT}) = \circ$
	↙ ↘		
$a(\mathcal{L}_{CONST}) = \circ$	S_3	S_4	$a(\mathcal{L}_{PRED}) = \circ$
	↘ ↙		
	S_ω		$a(\text{anything}) = \circ$

Figure 1 shows this as a subgraph of the lattice of theories that results from adjusting the *a* function. Each arrow in the figure adds a condition to *a*. S_1 is the theory of subject matters with no blanked out elements, S_2 is equivalent to the issue theory, S_3 is a theory of predicative subject matters, S_4 is a theory of objectual subject matters, and S_ω is a trivial theory that assigns the empty set as the subject matter of all sentences (this theory lies at the bottom of the lattice). For the reasons I have given already, we should not expect a theory below $S_{1.4}$ to be able to distinguish between counterfactuals and other conditionals, even if it could distinguish between counterpossibles (in the issue theory S_2, "if Hobbes had squared the circle, Hobbes would have squared the circle" and "if Hobbes had squared the circle, children in the Andes would have cared" have different subject matters). This suggests that the weakest theory we should adopt from this perspective must be stronger than $S_{1.4}$. Because any theory at least as strong as S_2 (so *a fortiori* any theory stronger than $S_{1.4}$) has the resources to account for most types of subject matter of interest, nothing seems to be lost in doing so.[35]

[34] Fine (1986) considers some variations on theories where connectives contribute to the content of sentences.

[35] Note that we need a theory at least as strong as S_4 to meet the third desideratum.

Thus, despite its closeness to syntax, the top theory S_1 may not be a bad candidate after all for the theory of prime subject matters, at least for our purposes here.[36]

The structure that results gives one suggestion about how subject matters in the sense of these theories can be mapped to questions. Pick a theory of subject matter S_n and take the prime subject σ_{Sn} for a sentence φ. We can get a set of questions by producing variations of σ_{Sn} where any number of nodes (either sub-sentential or sentential) are replaced by a series of indexed "null" elements $\_{}_1, \ldots, \_{}_n$. So, e,g., for a sentence *Fa* we have $\sigma_{S_1} = \{\langle \mathfrak{F}, \mathfrak{a} \rangle\}$, and then we get the set $Q = \{\langle \mathfrak{F}, \mathfrak{a} \rangle, \langle \_{}_1, \mathfrak{a} \rangle, \langle \mathfrak{F}, \_{}_1 \rangle, \langle \_{}_1, \_{}_2 \rangle, \_{}_1\}$. These structures can be interpreted as the questions "is *a F*?", "what is true of *a*?", "what does *F* apply to?", "what is true of what else?", and "what is true?". In the case of counterfactuals, we will have (among others) structures of the form $\langle \Box\!\!\rightarrow, s(\varphi), \_{}_1 \rangle$, which can be interpreted as standing for questions of the form "what would be true, if φ were true?". The same applies, *mutatis mutandis*, for counterpossibles. Thus, we can count our second desideratum (that required us to link the subject matter of counterpossibles to questions of precisely this form) as satisfied.

4. Counterfactuals and Patterns in Modal Spaces

There may be several objections to the approach of enriching atom-based accounts of subject matter in the way I have done in the previous section (in fact, in Section 5 I will raise an issue against it). Since in any case we need a theory of subject matters for counterfactuals and counterpossibles, it is desirable that objectors are able to provide a replacement. One broad class of objectors would be constituted by those who would prefer to follow a way-based approach to the construction of a theory of subject matters. Can we have a theory that meets all the desiderata if we start from a way-based approach?

One way to attempt it, as hinted above, would be to extend modal space with impossible worlds, and adopting an alternative semantics for counterfactuals.[37] Then, were could be worlds where counterpossibles are true or false in different ways (like in Fine's state semantics, where they would have different truth- and falsity-makers). To see how the approach would look like, consider Berto, French, Priest & Ripley's (2018) semantics for counterfactuals. They assume that there is a universe of worlds, both possible and impossible. In their semantics, frames include an accessibility relation R_φ for every formula φ in the language, and their models add to the frames a valuation function v that assigns truth values to sentences at worlds. For possible worlds, v assigns a value only to propositional parameters. At impossible worlds, the value of all sentences is assigned

[36] To capture less fine-grained theories, the s function could also be adjusted in order to make different sentences have the same subject matter despite syntactical differences. That way, for example, $\sim(\varphi \wedge \sim\psi)$ and $\varphi \rightarrow \psi$ and could be made subject-matter-equivalent without having to blank out structural elements.

[37] Here I will not deal with criticisms to this kind of approach, although they remain influential (e.g., Williamson, 2018).

atomically (the guiding thought there is that at impossible worlds the laws of logic are different, in any possible way—so the value of $\varphi\nabla\psi$ could be anything; cf. Priest, 2005, p. 16). The truth conditions for the counterfactual are stated as follows:

- $w \vDash \varphi \mathrel{\Box\!\!\rightarrow} \psi$ iff for all w' such that $wR_{\varphi}w'$, $w' \vDash \psi$

The relation R_{φ} is intended to be understood so that $wR_{\varphi}w'$ means that w' is, *ceteris paribus*, the same as w, except that φ is true in w'. Accordingly, it is supposed that if $wR_{\varphi}w'$, then $w' \vDash \varphi$, and if $w \vDash \varphi$, then $wR_{\varphi}w$. Take

12) If Hobbes had squared the circle, Hobbes would have squared the circle.

Then, consider a world w_1 *ceteris paribus* like ours, except at that world Hobbes squared the circle (this is an impossible world). In w_1, Hobbes squared the circle, and since w_1 was arbitrary, at all worlds *ceteris paribus* like ours where Hobbes squared the circle, Hobbes squared the circle. So the conditional comes out as true. Now consider

13) If Hobbes had squared the circle, children in the Netherlands would have cared.

Likewise, consider a world where Hobbes squared the circle. There is a world that is *ceteris paribus* like it, where children in the Netherlands care for Hobbes's result. But at the same time, there is a world *ceteris paribus* like it where children do not care for it. So the counterfactual comes out as false. Now, consider a world where children in the Netherlands had paid special attention to the issue of whether the circle could be squared, and a world *ceteris paribus* like it where Hobbes had squared the circle. In worlds like that, we should expect children in the Netherlands to care about Hobbes' results. So in those worlds, the counterfactual should come out as true. In worlds where all European children are worried about squared circles, the counterfactual should also come out as true. Assuming Yablo's theory of subject matters, the subject matter of the counterfactual is the division of worlds where it is similarly true or false. The divisions whether (Dutch, European, any) children would have cared if Hobbes had squared the circle are included in the division corresponding to the subject matter what would happen if Hobbes had squared the circle. So the approach seems to be able to meet our first two desiderata. For meeting the third, we could stipulate that counterpossibles are about whatever is a part (in some sense) of all or some the ways in which the counterpossibles are true or false. So "if Hobbes had squared a circle, Hobbes would have squared a circle" is about Hobbes because it is a part of all the ways in which it could be true or false. I will revisit the point in Section 5.

It is worth considering a different way to extend the way-based approach. Underlying both Lewis' and Yablo's theories of subject matters is the idea that subject matters are "systems of differences" or "patterns of cross-world variation". In a cell-conception of subject matter like the one we have assumed so far, those patterns are "tilings" or "coverings" of logical space, and these are groupings of points (worlds, states) in logical space. Remember, however, the structure that Fine's proposal attributes to counterfactuals, with states having other states as outcomes. Rather than patterns of worlds/states, one might want to consider patterns of patterns of worlds/states: from states similar to the antecedents, we can get to states similar to the outcome-states of the antecedents. An idea, then, is to make the subject matters of counterfactuals patterns of transitions between states, which is naturally understood as a similarity relation over pairs of states and state-spheres. We do not need to assume that the relation between states and state-spheres is like Fine's outcome relation; a relation of similarity like in the traditional approach to counterfactuals could play a role here too. What would be true if I was not a philosopher is a system of different ways in which states in which I am not a philosopher relate to states that stand in certain relations to the states where I am not a philosopher. What would be true if I was not in Europe is part of patterns of patterns that also includes what would be true if I was not a philosopher, like what would be true if I was not in my actual situation. Intuitively, that is the pattern of patterns that includes all the counterfactuals about me (the common subject matter for all counterfactuals about me). To handle counterpossibles, we still need to include impossible worlds/states. But this extension to the way-based approach would yield an even finer-grained picture of subject matters.

5. Are Counterpossibles About the Things They Mention?

As we have seen, it is possible to develop theories of subject matters for counterpossibles along the lines of both atom-based and way-based approaches. These theories are in principle able to meet all the desiderata we set out for a theory of subject matters. Ideally, we should find a way to to decide between these approaches.[38] In this section I will raise an issue that could be decisive in this way. The problem is as follows. Suppose that it were reasonable to hold that a counterpossible is about what would happen if a fact concerning an object *a* were to happen at the same time that it is not about *a*. Under certain assumptions, the structurally-enriched atom-based theory of subject matters cannot not make this difference in a natural way, while the way-based theory can. Consequently, the latter should be preferred. The argument is not decisive because support for the crucial supposition is controversial. In what follows, I will try to motivate the supposition, show why the atom-based theory cannot make the distinction in a way that is natural, and how the way-based theory could.

[38] Why not think that both approaches give us valid accounts of subject matter? A form of pluralism could be tempting. However, even for a pluralist there might be a further question about which of the approaches is more fundamental. Hawke (2018) shows that a way-based conception of subject matter is derivative from the atom-based account that he endorses. But the fact that a way-based theory can be derived like this does not entail that atom-based approach is in general more basic.

To see what may motivate the supposition, consider a counter-mathematical:

14) If 1 + 1 was 3, then 1 + 2 would be 4.

It makes some sense, I think, to believe that (14) is about what would happen if 1 + 1 was 3.[39] This is what the CSP seems to predict, so one may be independently disposed to believe it. Does it make sense to think that (14) is about the numbers 1, 2, 3 and 4, that it mentions (that is, that {1, 2, 3, 4} is its objectual subject matter)? Now, I think that is something that one may find less intuitive, even if one thought that (14) must have some objectual subject matter.[40] This combination of positions may perhaps be held reasonably; so the supposition (of which this is an instance) may be *prima facie* plausible. I suspect a view where the supposition is vindicated will be attractive to those who are somewhat skeptical (but not fully skeptical) about counterpossibles, since it would allow for counterpossibles to have subject matters, while failing to be about certain contested subject matters.[41]

In a broader sense of "mentioning", we could want to block the inference that counterpossibles are about items of other syntactical classes. For example, assume that it is conceptually and logically necessary that in a disjunction, if one of the disjuncts is true, the disjunction is true.[42] Then, a disjunction being not-true if one of the disjuncts is true would constitute a logical impossibility. Consider, then, the counterpossible:

15) If there was a not-true disjunction φ with a true disjunct and ψ was true, the disjunction $\varphi \vee \psi$ would be not-true.

As per the CSP, this is about what would be true if there was a not true disjunction φ with a true disjunct, and something else was true. We could ask if this counterpossible could be in some sense about the property of being a disjunction.

[39] Note that this is not the same as being about a truth concerning what would happen if 1 + 1 was 3; the question of the truth value of the counterpossible is a separate issue.

[40] Remember Gendler's (2000) quote on what sentences describing impossibilities are about. More recently, Tump (2021) argues that numbers are given collectively in the context of number systems. In the case of natural numbers, they are characterized by the properties that are a consequence of the Peano axioms. Thus, we cannot change their relations to other numbers without also changing the number system. In those conditions, we cannot say that we are talking about the same numbers.

[41] A further form of skepticism could propose that even if we have reasons to think that counterpossibles have subject matters, we cannot know what they are.

[42] This fails for the disjunction in Weak Kleene Logic (WKL), which raises concerns about in what sense that logic has a disjunction (cf. Omori, Szmuc, 2017). One answer is that WKL has a disjunction in what respects to determinate values; so maybe what is a conceptual necessity about disjunction is that if one of the disjuncts is true and both disjuncts are determinate, the disjunction is true (cf. Beall, 2016 for some discussion on how to interpret WKL in terms of subject-sensitivity).

If we hold fixed as essential to the property that disjunctions with at least one true disjuncts are true, (15) cannot be about the property of being a disjunction.

This way to motivate the supposition faces two main objections. First, one may wonder: if the intuition against these counterpossibles to be about the things they mention in a broad sense holds, why does that not also undermine the CSP intuition? The point would be that counterpossibles could not be about what would happen if some impossibility φ concerning *a* occurred without also being about *a*. Second, one may think that linguistic items must be about the subject matters that they are intended to be about, so, since it it is implausible that these counterpossibles were not intended to be about the things they mention, they must be about those things.

The first objection is worrying because it seems to force us to say that the subject matter of counterpossibles is degenerate after all. However, we have enough resources to resist this concession. Plausibly, something that is about what would happen if some impossibility φ concerning *a* occurred (Q_1) is also about what would happen if some impossibility φ concerning *something* occurred (Q_2). That is, we can abstract from the identity of the things that counterpossibles seem to mention and still have a viable subject matter. After all, when we are worried about patterns, we are not always worried about the bearers of those patterns except in that they are bearers of those patterns. So one way to respond to the objection is that we only need to be able to say that counterpossibles are about this kind of thing, rather than what the naive version of the CSP says. A sophisticated version of the CSP could be more informative. There are questions that these counterpossibles answer to that are more committal to *a* than Q_2 without being as committal as Q_1. The questions that constitute the subject matter of the counterpossibles may not be as abstract as Q_2: plausibly, these counterpossibles are also about what would happen if some impossibility φ concerning *something relevantly similar to a* occurred.[43] This is, to be sure, about *a*, but not as directly as Q_1 is about *a*, because the relevant properties of *a* in each case are different (for Q_2, it only matters that *a* is similar to whatever the counterpossible is about, whereas for Q_1 it matters that *a* itself could somehow be in the conditions given in the antecedent). We want to avoid Q_1's way of being about *a*. This also gives a simple error theory that explains why someone may think that counterpossibles are about the things they mention: they may realize that they are about them in some way, but misidentify the way in which they are about them.

This response to the first objection is not sufficient to dismiss the second objection, since it may still be objectionable that our theory of subject matters does not vindicate what is intended to be the topic of a counterpossible as its topic. If someone puts fort a counterpossible mentioning *X* with the intention to talk about *X*, why should we not believe that the counterpossible is about *X*? In re-

[43] This does not necessarily make the antecedents of counterpossibles to be about possibilities. It might still be impossible that anything similar to *a* has the properties that are attributed to it there.

sponse, we could argue that merely intending to talk about something does not guarantee that one talks about it; one might, for example, have mistaken the topic that is under discussion.[44] Whether there is a mismatch between what counterpossibles are about and what they are intended to be about is not decisive against the claim that counterpossibles are not about the things the seem to mention.[45]

The supposition seems initially defensible, then, after some adjustments. Let us consider if the approaches to the subject matter of counterpossibles I have sketched can respect it.

The enriched atom-based account immediately falls into difficulties. Because the subject matter of counterpossibles is constructed from the subject matter of their constituent atoms, and the way that predicates and constants are assigned concepts is direct, counterpossibles will turn out to be about the things that they mention. Thus, e.g., (14) will have as its prime subject matter something like

$$\{\langle \Box\!\!\rightarrow, \{\langle \mathfrak{Eq}, \langle \mathfrak{Sum}, 1, 1\rangle, 3\rangle\}, \{\langle \mathfrak{Eq}, \langle \mathfrak{Sum}, 1, 2\rangle, 4\rangle\}\rangle\}$$

from which we can recover the objectual subject matter $\{1, 2, 3, 4\}$. It is not obvious how this result can be prevented. One way would be to adjust the assignment function, perhaps making it assign concepts that correspond to the idea of something that is similar to 1, 2, 3 and 4 in the relevant way (let us say, 1’, 2’, 3’, and 4’). Then, we would get that the subject matter of (14) would be

$$\{\langle \Box\!\!\rightarrow, \{\langle \mathfrak{Eq}, \langle \mathfrak{Sum}, 1', 1'\rangle, 3'\rangle\}, \{\langle \mathfrak{Eq}, \langle \mathfrak{Sum}, 1', 2'\rangle, 4'\rangle\}\rangle\}$$

How to make it so the theory makes these adjustments only in the case of counterpossibles (and perhaps other expressions like statements concerning impossibilities)? It seems clear that the atom-theorist has to make the assignment function sensitive to the semantics of the atoms (in general, we cannot distinguish counterpossibles by purely syntactic means). However, this is undesirable if one wants the theory of subject matters to be independent from considerations

[44] This is related to the issues that Munro and Strohminger (2021) raise concerning the idea that the contents of imaginings are simply determined by whatever contents one intends to imagine (and of course, their argument there applies to attempts to imagine a counterpossible as true as well), a position they call Intentionalism. Interestingly, Intentionalism is a substantial assumption in Berto and Schoonen’s (2017) approach to imagination; the authors make use of Kung’s (2016) idea that part of the content of imaginings is stipulated to argue that impossibilities can be imagined. The question I am raising here can be understood as whether aboutness properties can be stipulated or not.

[45] A fuller answer would have to address the issue of what function counterpossible-talk serves, in order to examine whether counterpossible talk indeed requires intentions to talk about the items that they mention, but I will not dwell on this here.

about the truth conditions of sentences, and many atom-theorists impose this restriction upon themselves.[46]

The way-based approach exhibits no such qualms about the independence of subject matter and truth conditions to begin with, so it is better positioned to deal with the issue. The manner in which a ways-theorist would filter out objectual aboutness for counterpossibles is similar to what the proponent of the enriched atom-based theory has available: first, we check if we are dealing with a structure with an impossible antecedent; then, we abstract from the counterpossibles so that the elements in questions are neutered (essentially, taking the focus of the counterpossible explicitly away from its putative referring terms); and finally, we evaluate the subject matter of the resulting structure.

There are, then, some reasons to prefer a way-based approach to an atom-based approach, at least in the case of counterpossibles. However, as I said before, these reasons are not decisive, for two reasons: first, because they depend on certain controversial intuition-based assumptions about what we can say about the subject matter of counterpossibles, and second, because proponents of the atom-based approach have a way to deal with the issue, namely, dropping the assumption that subject matter is independent from truth conditions. Whether that is too costly for such theorists is not my concern here, although the dialectical situation suggests that we may not be able to bypass the issue of the semantics of counterpossibles after all. For those who do not share that assumption, another alternative could be to adopt some form of pluralism about subject matters and make use of an overall theory that combines the insights of both way-based and atom-based approaches. Such a theory is not yet available.

REFERENCES

Badura, C. (2020). More Aboutness in Imagination. *Journal of Philosophical Logic*, 50(3), 523–547.

Baker, A. (2007). A Counter on Counterpossibles. *The Reasoner, 1*(2), 7–8.

Barwise, J. (1989). *The Situation in Logic* (Lecture Notes 17). Stanford: CSLI.

Barwise, J., Perry, J. (1983). *Situations and Attitudes*. Cambridge: The MIT Press.

Beall, JC. (2016). Off-Topic: A New Interpretation of Weak Kleene Logic. *The Australasian Journal of Logic, 13*(6), 136–142.

[46] Hawke (2018, p. 26), for example, argues in favor of the issue-theory over the Finean state-based theory of subject matter on the grounds that the former,but not the latter, can make certain distinctions about the subject matter of problematic sentences without having to appeal to what makes those sentences true or false. Yablo (2014, p. 2) suggests that subject matter is constrained, but not determined by truth conditions. Plebani and Spolaore (2020, p. 15) point out that in their account of subject matter, this Yablovian condition is respected, but also there is no way to recover the truth conditions of a sentence from its SM, which they take as a positive point for their account (cf. also Berto, Hawke, Hornischer, 2019).

Bernstein, S. (2016). Omission Impossible. *Philosophical Studies*, *173*(10), 2575–2589.

Berto, F. (2018). The Theory of Topic-Sensitive Intentional Modals. In I. Sedlár, M. Blicha (Eds.), *The Logica Yearbook* (pp. 31–56). College Publications.

Berto, F., French, R., Priest, G., Ripley, D. (2018). Williamson on Counterpossibles. *Journal of Philosophical Logic, 47*(4), 693–713.

Berto, F., Hawke, P., Hornischer, L. (2019). *Foundations of Two-Component Semantics*. Manuscript in preparation.

Berto, F., Jago, M. (2019). *Impossible Worlds*. Oxford: Oxford University Press.

Berto, F., Özgün, A. (2021). Indicative Conditionals: Probabilities and Relevance. *Philosophical Studies*. doi:10.1007/s11098-021-01622-3

Berto, F., Schoonen, T. (2017). Conceivability and Possibility: Some Dilemmas for Humeans. *Synthese*, *195*(6), 2697–2715.

Brogaard, B., Salerno, J. (2013). Remarks on Counterpossibles. *Synthese*, *190*, 639–660.

Edgington, D. (2008). Counterfactuals. *Proceedings of the Aristotelian Society*, *108*, 1–21.

Emery, N., Hill, C. (2016). Impossible Worlds and Metaphysical Explanation: Comments on Kment's 'Modality and Explanatory Reasoning'. *Analysis*, *77*(1), 134–148.

Fine, K. (1986). Analytic Implication. *Notre Dame Journal of Formal Logic*, *27*(2), 169–179.

Fine, K. (2012). Counterfactuals Without Possible Worlds. *The Journal of Philosophy*, *109*(3), 221–246.

Fine, K. (2016). Angellic Content. *Journal of Philosophical Logic, 45*, 199–226.

Fine, K. (2020). Yablo on Subject-Matter. *Philosophical Studies, 177*, 129–171.

Fine, K. (2021). Constructing the Impossible. In L. Walters, J. Hawthorne (Eds.), *Conditionals, Paradox and Probability: Themes from the Philosophy of Dorothy Edgington* (pp. 141–163). Oxford: Oxford University Press.

Gendler, T. (2000). The Puzzle of Imaginative Resistance. *The Journal of Philosophy, 97*(2), 55–81.

Hawke, P. (2018). Theories of Aboutness. *Australasian Journal of Philosophy*, *96*(4), 697–723.

Jago, M. (2014). *The Impossible: An Essay on Hyperintensionality*. Oxford: Oxford University Press.

Kim, S. Maslen, C. (2006). Counterfactuals as Short Stories. *Philosophical Studies*, *129*, 81–117.

Kment, B. (2006). Counterfactuals and the Analysis of Necessity. *Philosophical Perspectives*, *20*(1), 237–302.

Kment, B. (2014). *Modality and Explanatory Reasoning*. Oxford: Oxford University Press.

Kment, B. (2016). Replies to Dorr, Emery and Hill and Yablo. *Analysis*, *77*(1), 166–188.

Kocurek, A. W. (2018). Counteridenticals. *The Philosophical Review*, *127*(3), 323–369.

Kocurek, A. W., Jerzak, E. J. (2021). Counterlogicals as Counterconventionals. *Journal of Philosophical Logic*, *50*(4), 673–704.

Kung, P. (2014). You Really Do Imagine It: Against Error Theories of Imagination. *Noûs*, *50*(1), 90–120.

Leitgeb, H. (2018). HYPE: A System of Hyperintensional Logic (with an Application to Semantic Paradoxes). *Journal of Philosophical Logic*, *48*(2), 305–405.

Lewis, D. K. (1973). *Counterfactuals*. Oxford: Blackwell.

Lewis, D. K. (1998a). Relevant Implication. In D. Lewis, *Papers in Philosophical Logic* (pp. 125–155). Cambridge: Cambridge University Press.

Lewis, D. K. (1998b). Statements Partly about Observation. In D. Lewis, *Papers in Philosophical Logic* (pp. 111–124). Cambridge: Cambridge University Press.

Locke, T. (2019). Counterpossibles for Modal Normativists. *Synthese*. doi:10.1007/s11229-019-02103-1

Munro, D., Strohminger, M. (2021). Are We Free to Imagine What We Choose? *Synthese*, *199*, 1–18.

Nolan, D. (1997). Impossible Worlds: A Modest Approach. *Notre Dame Journal of Formal Logic, 38*, 535–572.

Omori, H., Szmuc, D. (2017). Conjunction and Disjunction in Infectious Logics. *Lecture Notes in Computer Science, 10455*, 268–283.

Perry, J. (1986). Possible Worlds and Subject Matter. In J. Perry, *The Problem of the Essential Indexical and Other Esssays* (pp. 145–160). Palo Alto, CA: CSLI Publications.

Plebani, M., Spolaore, G. (2020). Subject Matter: A Modest Proposal. *The Philosophical Quarterly*. doi:10.1093/pq/pqaa054

Priest, G. (2005). *Towards Non Being: The Logic and Metaphysics of Intentionality*. Oxford: Clarendon Press.

Priest, G. (2016). Thinking the Impossible. *Philosophical Studies*, *173*(10), 2649–2662.

Ryle, G. (1933). About. *Analysis*, *1*(1), 10–12.

Stalnaker, R. (1968). A Theory of Conditionals. In N. Rescher (Ed.), *Studies in Logical Theory* (American Philosophical Quarterly Monographs 2, pp. 98–112). Oxford: Blackwell.

Stalnaker, R. (1987). *Inquiry*. Cambridge, MA: MIT Press.

Starr, W. B. (2014). A Uniform Theory of Conditionals. *Journal of Philosophical Logic, 43*(6), 1019–1064.

Tan, P. (2019). Counterpossible Non-Vacuity in Scientific Practice. *Journal of Philosophy*, *116*(1), 32–60.

Tump, L. (2021). Mathematical Counterfactuals with Number-Theoretic Antecedents and Extra-Mathematical Explanation. *Logique Et Analyse, 254*, 191–213.

Vander Laan, D. (2004). Counterpossibles and Similarity. In F. Jackson, G. Priest (Eds.), *Lewisian Themes: The Philosophy of David K. Lewis* (pp. 258–275). Oxford: Clarendon Press.

Vetter, B. (2016). Counterpossibles (Not Only) for Dispositionalists. *Philosophical Studies, 173*(10), 2681–2700.
Weiss, Y. (2019). *Frontiers of Conditional Logic*. (Unpublished PhD thesis). The Graduate Center, City University of New York (CUNY), New York.
Williamson, T. (2007). *The Philosophy of Philosophy*. Oxford: Blackwell Publishing.
Williamson, T. (2018). Counterpossibles. *Topoi*, *37*, 357–368.
Williamson, T. (2021) Degrees of Freedom: Is Good Philosophy Bad Science? *Disputatio*, *13*(61), 73–94.
Yablo, S. (2014). *Aboutness*. Princeton, NJ: Princeton University Press.
Yablo, S. (2018). Reply to Fine on Aboutness. *Philosophical Studies*, *175*(6), 1495–1512.
Yablo, S. (2020) Nonexistence and Aboutness: The Bandersnatchers of Dubuque. *Critica*, *52*(154), 77–100.
Yagisawa, T. (2010). *Worlds and Individuals, Possible and Otherwise*. Oxford: Oxford University Press.