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# SYNTAX–SEMANTICS INTERACTION IN MATHEMATICS**

Summary: Mathematical tools of category theory are employed to study the syntax-semantics problem in the philosophy of mathematics. Every category has its internal logic, and if this logic is sufficiently rich, a given category provides semantics for a certain formal theory and, *vice versa*, for each (suitably defined) formal theory one can construct a category, providing a semantics for it. There exists a pair of adjoint functors, Lang and Syn, between a category (belonging to a certain class of categories) and a category of theories. These functors describe, in a formal way, mutual dependencies between the syntactical structure of a formal theory and the internal logic of its semantics. Bell's program to regard the world of topoi as the *univers de discours* of mathematics and as a tool of its local interpretation, is extended to a collection of categories and all functors between them, called "categorical field". This informal idea serves to study the interaction between syntax and semantics of mathematical theories, in an analogy to functors Lang and Syn. With the help of these concepts, the role of Gödel-like limitations in the categorical field is briefly discussed. Some suggestions are made concerning the syntax-semantics interaction as far as physical theories are concerned.

Key words: philosophy of mathematics, categorical logic, syntax-semantic interaction, Bell's program, Gödel-like limitations.

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## 1 Introduction

Mathematics is what mathematicians are doing when they do their work as mathematicians. They create new formalisms, formulate axioms, deduce theorems. This is more or less clear. It seems that for a philosopher who decides to interfere into the mathematicians' work, it is left only to determine syntactical rules and reconstruct everything in a logical order. However, there are important questions involved in this kind of ordering work. What mathematics is about? How do exist mathematical formalisms? Have axioms and theorems some kind of independent existence? In other words, how to cope with the semantics of mathematics? This is a vast terrain open for philosophers of mathematics.

There is an opinion that "mathematics is closed with respect to its philosophy", i.e. that it is possible to do philosophy of mathematics in a mathematical way. In last decades, mathematical category theory was the arena of enormous progress, and at the same time became a powerful tool for many philosophical investigations (see, for instance, Landry [2017]). Its impact on the philosophy of mathematics was evident almost from the very beginning. In the present work, I employ mathematical tools of this theory to face the syntax-semantics problem in the philosophy of mathematics.

It is a standard way to logically organize a mathematical theory, a theory $T$, say, so as to have its syntax well defined. It is also well known that every category, for instance a category $C$, has its internal logic, and if this logic is sufficiently rich, the category $C$ provides semantics for the theory $T$. If this is the case, we can say that $T$ is about $C$. Moreover, the categorical logic allows us to investigate the interaction between syntax and semantics of $T$. More precisely, there exists a pair of functors, Lang and Syn, from a category, belonging to a certain class of categories (for instance, coherent categories) to a category of theories, and *vice versa*, which mathematically describe this interaction. Moreover, it turns out that these functors are adjoint functors. Land and Syn describe, in a formal way, mutual dependencies between the syntactical structure of $T$ and the internal logic of its semantics. Syntax and semantics are interwoven with each other in a manner corresponding to the adjointness of the functors Lang and Syn.

This is presented in sections 2 and 3. They contain material well known to category theorists, and the style of presentation is didactic rather than mathematically precise. For the sake of simplicity, the analyses are, in principle, restricted to the first order predicate logic. Section 2 focuses on preliminaries; section 3 penetrates more deeply into the role of functors Lang and Syn.

We are now ready to make category theory operational in the domain of the philosophy of mathematics. It was John Bell (1986) who proposed to abandon set-theoretic approach to foundations of mathematics, and to regard topos theory as the *univers de discours* for mathematics. Consequently, according to him, mathematical concepts have only local meaning (i.e. only with respect to a topos equipped with a natural numbers object [NNO], called local framework), and the truth values of mathematical statements are defined only locally (only those mathematical assertions have absolute truth values that are invariant with respect to admissible transformations between local frameworks). My proposal, rather a loose idea, is to extend Bell's program beyond the realm of topoi to the conglomerate (certainly not a category) of all categories and all functors between them. To underline an informal and indeterminate character of this conglomerate, I propose to call it "categorical *field*". Since categories have their internal logics, logic is a "local variable" in the categorical field. To the field of categories there corresponds the "field of theories", and interactions between their syntactic and semantic aspects also develop locally. Some people claim that Gödel's theorems caused crisis in mathematics, but the theorems themselves and their consequences are valid only in those categories which contain basic arithmetic, e.g. Peano's arithmetic. In the categorical field, the "crisis" has certainly only a local outreach.

This is covered in sections 4 and 5. Section 4 focuses on Bell's program and its extension to the categorical field; section 5 deals with Gödelian limitations of mathematics.

After reflecting on the nature of mathematics, the natural question is: what about physical theories? As far as they employ mathematical theories, they are subject to the same syntactic and semantical rules; the essentially new aspect is their reference to the physical world. This domain or aspect of the physical world, to which a given physical theory $T$ refers, I propose to call natural semantics ($NS$) of $T$. It is tempting to consider two "adjoint functors", I call them Mat and Measur, between natural semantics and physical theories, and *vice versa*, in a loose analogy with functors Lang and Syn, and use them to investigate interactions between syntactical structures of physical theories and those domains or aspects of the physical world these theories are about.

This is done in section 6. It is obvious that in this section, one must go beyond the strict formalism.

## 2 Categories and Logic

In this section, I start to present this material from category theory and categorical logic that is indispensable to grasp the interplay between syntax and semantics of formal theories.[1] We must first make precise what do we mean by a formal theory. The definition must be strict to enable formal manipulations, and at the same time broad enough to embrace real mathematical theories. To ensure the latter, we will assume that it is a type theory, i.e. that each of its terms is equipped with a specific type. Many mathematical theories use a single type language but category theory, for instance, uses either two type language (with objects and morphisms as types) or a single type language (with morphisms as a type). A rule assigning a type to a term is called a type assertion (for details about type, terms and formulae see Appendix). To guarantee a sufficiently broad character of the approach, we will assume that the theory in question is an axiomatic theory.

**Definition 1** *A (type) theory T consists of:*

1. *a set S of types,*
2. *a set V of variables with a type assigned to each variable,*
3. *a set F of function symbols with a type assigned to each domain and codomain of every function symbol,*
4. *a set R of relation symbols with a type assigned to each argument of every relation symbol,*
5. *a set of logical symbols,*
6. *a set A of axioms (in the form adapted to the type formalism).*

[1] In doing so I essentially follow Fu (2015) with some simplifications. I also recommend reading Bell (2017).

The relation symbols may also refer to unary relations (i.e. relations having one argument); we may interpret them as properties (for details of this definition see Mac Lane, Moerdijk [1992, p. 527]). For instance, the Zermelo-Fraenkel set theory (with the axiom of choice) can be put into this form. Let us notice that not only mathematical theories can be formulated as type theories.

A sufficiently rich category $C$ with finite limits can be made (in a canonical way) a model for any type theory $T$. In this way, $C$ provides a semantics for $T$, the so-called categorical semantics.

**Definition 2** *The categorical semantics* $[\![\cdot]\!]$ *for a theory* $T$ *is defined in the following way:*

1. *for each type* $A$, $[\![A]\!]$ *is an object in the category* $C$,
2. *for each function symbol* $\boldsymbol{f}$ *with the types* $A$ *and* $B$ *for its domain and codomain, correspondingly,* $[\![\boldsymbol{f}]\!]$ *is a morphism* $[\![A]\!] \longrightarrow [\![B]\!]$ *in* $C$,
3. *for each relation symbol* $R$ *of a type* $A$ *(with arguments of certain types) and a term* $t$, $[\![R(t)]\!]$ *is a chosen subobject* $[\![R]\!]$ *of* $[\![A]\!]$ *in* $C$.[2]

To this definition we must add all of (first order) logic which is used by $T$ expressed in terms of $[\![\cdot]\!]$, but this is almost obvious and going deeper into this matter is not necessary for the rest of my argument (for details see Fu [2015]).

What does this mean "sufficiently rich category"? Categories have their own "internal logics", and if such a logic is too weak, it is unable to provide semantics for the theory $T$. Any category $C$ with final limits can model a logic with operators $\vee$ and $\top$, but for a stronger logic more structure would be required. Usually, it is enough to consider theories formulated in the so-called coherent logic. This is a part of the first order logic using only the connectives $\wedge$ (and) and $\vee$ (or), $\top$ (true), $\bot$ (false), and the existential quantifier (for a full definition see *Coherent Logic* [2018]). A category, the internal logic of which is coherent is called a coherent category. In the coherent logic there is no difference between classical and intuitionistic logic. Moreover, large parts of mathematics can be axiomatized as coherent theories.

Let us make the concept of the internal logic for a category $C$ precise.

[2] Technically, this means that for every relation $R$ and any term $t$, $R(t)$ is the pullback of $R$ along $t$ (for details see Low [2013]).

**Definition 3** *The internal logic of a category C is defined in the following way:*

1. *its types are objects of C,*
2. *its variables are identity morphisms in C,*
3. *its function symbols are non-identity morphisms in C,*
4. *its relation symbols are subobjects in C. If $\varphi$ is a subobject of an object A in C, it can be regarded as a proposition; by analogy with the usual set theory, just think of $\varphi$ as a collection of all things of type A which verify $\varphi$.*

*Logical symbols are assumed as usual, and if the category C is to be regarded as a semantics for a theory T, axioms of T must be given by the order relation on the subobject poset in C.*[3]

In the obvious way, internal logic of a category $C$ defines a type theory $T$ for which $C$ provides the semantics. Therefore, if a statement is provable in $T$, it is also true in $C$ (soundness), and *vice versa* if a statement is provable in $C$, it is true in $T$ (completeness). The fact that we can "extract a type theory out of any category" (Fu, 2015) allows us to treat definition (3) as defining a functor, call it Lang, from categories to type theories. To change definition (3) into the formal definition of the functor Lang we need only to determine a suitable category of categories and a suitable category of theories between which this functor operates. If this is done, given a category $C$ of CATEGORIES, we simply identify Lang($C$) in THEORIES with the internal logic of $C$ as it is defined in definition (3).

Let us first define the category THEORIES of theories. As it should be expected, its objects are theories and, less obviously, there is an arrow from a theory $T$ to a theory $T'$, $T \longrightarrow T'$, if one can express (interpret) $T$ in terms of $T'$ (Fu, 2015).

We also define the category CATEGORIES of categories as the collection of those categories (with corresponding functors as morphisms) that have enough structure required by definition (3) to provide semantics for theories of THEORIES (for details see: Fu [2015], *Internal Logic* [2018]). To make this rather sketchy description more concrete, we may agree to identify the category CATEGORIES with the category of coherent categories having coherent functors as morphisms.

[3] A proposition $\varphi$ implies a proposition $\psi$ if, regarded as subobjects of an object $A$ in $C$, they are connected by a morphism $\varphi \hookrightarrow \psi$ in the poset of subobjects of $A$.

In this way, definition (3) determines the functor Lang: CATEGORIES → THEORIES. The nice thing is that we also have a functor "in the reverse direction", Syn: THEORIES → CATEGORIES. If $T \in$ THEORIES, then Syn($T$) is called the syntactic category (of $T$). Before we define it, we should introduce some new terminology.

The pair ($\Gamma$,$\Phi$), where $\Gamma$ is a collection of type assertions[4] and $\Phi$ a collection of well defined formulae, is called a context. It is a formalization of what in ordinary language we mean by this term. In the present case, it consists of everything that has to be assumed to render a given assertion valid. When $T$ is a mathematical theory, this practically means to explicitly determine types of all its terms.

**Definition 4** *Let T be a type theory. Its syntactic category Syn(T) is defined in the following way:*

1. *its objects are contexts* ($\Gamma$,$\Phi$),
2. *its morphisms* ($\Gamma$,$\Phi$) → ($\Delta$,$\Psi$) *are interpretations of variables, i.e. for each type, prescribed by* $\Delta$*, we must be able to construct an element of this type out of data contained in* $\Gamma$. *We also must, for each assumption required by* $\Delta$ *(if there are any) present a proof of this assumption out of assumptions contained in* $\Gamma$.[5]

Syn($T$) is also called a category of contexts (for details and examples see *Syntactic Category* [2018]). From a theory $T$ we have constructed the category Syn($T$). For all practical purposes $T$ and Syn($T$) are the same, but Syn($T$) is less dependent on the language than $T$, in the following sense: let $T$ and $T'$ be two type theories having different structures (therefore, they differ as far as the language is concerned), but Syn and Syn($T$) may be equivalent as categories. If the latter is the case, the theories $T$ and $T'$ are said to be Morita equivalent.[6] Usually, theories

[4] For instance, $\Gamma = \{x_1{:}A_1,\ldots,x_n{:}A_n\}$ assigns types $A_1,\ldots,A_n$ to variables $x_1,\ldots,x_n$, respectively.

[5] More precisely, if $\Gamma = \{x_1{:}A_1,\ldots,x_n{:}A_n\}$ and $\Delta = \{y_1{:}B_1,\ldots,y_n{:}B_n\}$ are two contexts, then a morphism ($\Gamma$,$\Phi$) → ($\Delta$,$\Psi$) is a collection of terms $\Gamma \vdash t_1{:}B_1,\ldots,\ \Gamma \vdash t_n{:}B_n$. This means that to give this morphism we must give, for each type, or assumption, required by $\Delta$, a method to construct an element of that type, or a proof of that assumption, from the data or assumptions given by $\Gamma$.

[6] Morita equivalence is usually defined without mentioning syntactic categories; for instance, two theories are said to be Morita equivalent just in case their classifying toposes are equivalent (Tsementzis, 2015). Originally, Morita equiva-

are regarded just equivalent if they are Morita equivalent (for details and discussion see Halvorson, Tsementzis [2017]). In such a case, one defines a morphism between theories $T$ and $T'$ to be a functor between their syntactic categories $\mathrm{Syn}(T) \to \mathrm{Syn}(T')$. "Moreover, the fact that the syntactic category is defined »syntactically« means that a morphism $T \to T'$ actually induces a »translation« of the types, functions, and relations of $T$ into those of $T'$" (*Internal Logic*, 2018).

It turns out that the functors Syn and Lang are adjoint functors. Since the object $\mathrm{Syn}(T)$ in CATEGORIES provides a semantics for the theory $T$, and the object $\mathrm{Lang}(C)$ in THEORIES provides a syntax for the category $C$, the adjointness of these two functors determines a strict interaction between semantics and syntax. This can be seen from the two following formulae

(1) $\mathrm{Syn}(\mathrm{Lang}(C)) \to C$,

(2) $T \to \mathrm{Lang}(\mathrm{Syn}(T))$,

with both morphisms natural (in the sense of category theory), which formally express the adjointness of Syn and Lang functors.

As these two formulae are heavy with meaning, they call for a deeper attention.

## 3 Theories and Functors

Let us start with formula (1). We construct, by following instructions contained in definition 3, the internal logic of a category $C$, that is to say a theory $T_C = \mathrm{Lang}(C)$ for which $C$ provides a semantics. The objects of $C$ are types, its identity morphisms are variables, etc. But the theory $T_C$ should also satisfy some axioms. We inductively define an interpretation of each term (as a morphism in $C$) which may be constructed in $T_C$, and then we define (also inductively) an interpretation of each formula (as a subobject in $C$), which may be constructed in $C$ (see *Internal Logic* [2018]).[7]

lence was defined in the algebraic ring theory: two associative rings $R$ and $S$ with units are Morita equivalent if the category of (left) modules over $R$ and the category of (left) modules over $S$ are equivalent.

[7] For instance, in the case of group theory, there exist two arrows $G \times G \times G \to G$, which are interpretations of the following terms: $m(m(x,y),z)$ and $m(x,m(y,z))$ (the law of associativity).

The category $C$ contains logical tools able to express various axiomatics for the theory $T_C$; in other words, it potentially contains all possible axiomatics for $T_C$. The internal logic Lang($C$) is not only a way to describe the internal structure of $C$, but it also provides tools to prove (by "internal reasoning" in $C$) things in $C$ that are provable in $T_C$ (this is a content of the soundness theorem). The idea is to start with the axioms of a given type theory and make deductions from them by using standard methods, "which in practice amounts to pretending that the types are sets, the function symbols are functions, and the relation symbols are subsets", then we are entitled to conclude that "anything we prove will still be true when the theory is interpreted in an arbitrary category" (*Internal Logic*, 2018). Obviously, one is allowed to employ only those logical rules that are permitted by the logic appropriate to a given category; for instance, if we are working with a coherent theory, we must use rules of coherent logic.

We now construct the category Syn(Lang($C$)) = Syn($T_C$). It is a category whose objects are contexts, and morphisms are interpretations. Formula (1) tells us that "there is always a canonical model of the internal logic of $C$ within $C$" (Fu, 2015). We can abbreviate Syn($T_C$) to $C_{T_C}$. What can be proved in $C_T$, is true in all models of $T$ (Fu, 2015, Theorem 3.2).

Similar analysis can be carried out starting from formula (2), which can be written as $T \to T_{C_T}$. It turns out that the theory $T_{C_T}$ is Morita equivalent to $T$ (Tsementzis, 2015).

The above analysis shows a close interaction between syntax and semantics. This can intuitively be seen in natural languages, but here – for formal languages, in particular for formal mathematical languages – this is put into a nice formal interplay. It has the form of adjoint functors. This means that the morphism $C \to \text{Syn}(T)$ corresponds to a morphism $\text{Lang}(C) \to T$, and this correspondence is an isomorphism which is natural in $C \in$ CATEGORIES and $T \in$ THEORIES. This is written in the form

(3) $\quad \text{Hom}_{\text{CATEGORIES}}(C, \text{Syn}(T)) \cong \text{Hom}_{\text{THEORIES}}(\text{Lang}(C), T).$

The natural isomorphism $\cong$ plays an essential role in any adjunction situation.[8] It says that when $C$ varies in CATEGORIES and $T$ varies

[8] For a full definition of adjoint functors see any textbook on category theory.

in THEORIES, the isomorphism between morphisms Lang($C$) $\rightarrow T$ in THEORIES and $C \rightarrow$ Syn($T$) in CATEGORIES varies in a way that is compatible with composition of morphisms in CATEGORIES and THEORIES, correspondingly, and with the actions of Lang and Syn (Leinster, 2014). If we have in mind the meaning of functors Lang and Syn, we can see how intimately semantics and syntax interact with each other.

Formula (3) preserves its validity if theory $T$ is everywhere replaced by a theory $T'$ Morita equivalent to $T$. This means that a given theory can be expressed in various languages without changing its essential content.

Traditionally, one sharply distinguishes syntax and semantics. "This distinction is at the same time an epistemological and an ontological distinction, or at least it is motivated by epistemological and related ontological elements" (Marquis, 2010, p. 234). Syntax represents "the »concrete« facets of language and its epistemic accessibility is a crucial element", whereas semantics "is the world of »entities« syntactical expressions are supposed to refer to" (ibid.). Category theoretical approach fully respects this distinction, but is at the same time able to show nuances of their interactions with each other that were transparent to traditional logical tools. Loosely speaking, we not only have dependencies going from semantics to syntax, and *vice versa*, but since Lang and Syn are adjoint functors, these dependencies determine each other up to a unique natural equivalence.

## 4 The Extended Bell's Program

Material presented in the previous sections invites mathematically informed thinkers to draw some philosophical implications from it. Exactly this material inspired John Bell to develop a new approach to the philosophy of mathematics (Bell, 1981, 1986). "The fundamental idea is to abandon the unique absolute universe of sets central to the orthodox set-theoretic account of foundations of mathematics, replacing it by a plurality of local mathematical frameworks – *elementary toposes* – defined in category-theoretical terms" (Bell, 1986). Consequently, mathematical concepts lose their absolute meaning, and mathematical assertions their absolute truth values. Their meanings and truth values are defined locally, with respect to local frameworks, where by a local framework Bell understands a topos equipped with a natural numbers object (commonly abbreviated to NNO). When one changes from one local framework to another local framework, the meanings of concepts and truth values of assertions change accordingly. Formally, a change from one local framework to another is done with the help of what Bell calls admissible transformations. If $E$ and $F$ are local frameworks, an admissible transformation $f : E \to F$ is a pair of adjoint functors $f^*: E \to F$ and $f_* : F \to E$.

The analogy of this "philosophy" with that of the theory of relativity is evident. And as in special relativity, statements that are invariant with respect to Lorentz transformations are regarded as physical laws, statements that are invariant with respect to admissible transformations are regarded as mathematical laws. Since the internal logic of topoi is intuitionistic logic, "the invariant mathematical laws are those which are demonstrable *constructively*" (Bell, 1986).

Such a fundamental concept as that of real number is an example of a concept that changes depending on a local framework. Let $f : S \to \mathrm{Sh}(X)$ be an admissible transformation from a topos $S$ to the topos $\mathrm{Sh}(X)$ of sheaves on a topological space $X$ (both these topoi are local frameworks). What from the point of view of $\mathrm{Sh}(X)$ is a "real number", from the point of view of $S$ is a continuous real valued function on $X$, and the viewpoint is changing with the help of $f$.

Of course, in this game logic is involved from the very beginning. Topoi are the models for theories that are formulated in many-typed languages. "Each topos **E** is associated with such a language whose types match the objects of **E** and whose function symbols match the arrows of **E**. A theory in such a language is a set of sentences closed under intuitionistically valid deductions" (Bell, 1986). As we can see, our analysis, made in previous sections, fully applies to this case. And precisely here there appears a possibility to generalise Bell's program.[9] The limitation to topoi is justified only as far as one is concerned with details (such as local frames and admissible transformations), making the program concrete and possibly workable, and this was precisely Bell's goal. However, if one is concerned with the philosophy of mathematics as such, this limitation is neither wanted, nor useful.

[9] I call it a program, although, as far as I know, it never went beyond the original formulation.

It seems that Bell included the existence of NNO into the definition of a local framework since one could hardly imagine a mathematical theory without a counterpart of natural numbers, i.e. without something with the help of which one could count things. But even among topoi there are some in which "counting" is done with the help of an object different from NNO. For instance, in the so-called Zariski topos and Basel topos, in spite of the fact that the NNO does exist in them, the role of natural numbers is better fulfilled by the so-called smooth numbers (Moerdijk, Reyes, 2010, pp. 252, 289).[10] Therefore, in general, the insistence on the existence of NNO does not seem necessary.

Moreover, there are interesting domains of mathematics whose logic goes beyond intuitionistic logic, and consequently beyond the topos theory. If one wants to remain within a relatively well explored territory, one should mention the cotopos theory with its paraconsistent (or inconsistency-tolerant) logic. It is constructed by dualizing the usual topos theory (Angot-Pellissier, 2015; Estrada-González, 2014).[11] As it is well known, the algebra of open sets of a topological space (with negation of a proposition $p$ defined as the interior of the complement of an open set corresponding to $p$[12]) is a model of intuitionistic logic, and generates a Heyting algebra. By dualizing this construction, we obtain the algebra of closed sets of a topological space (with negation defined as the closure of the complement of a given closed set) which is a model of the paraconsistent logic. It generates a co-Heyting algebra, called also Brouwer algebra.[13]

Mathematical theories, which can find their semantics in cotopoi, are not necessarily some exotic mathematical structures. For instance,

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[10] In these topoi, when one uses NNO, the interval $[0,1]$ of $\mathbb{R}$ is non-compact and the object $R$ is non-Archimedean (here $R = C^{\infty}(\mathbb{R})$; $R$ is non-Archimedean, if the following axiom is not satisfied: $\forall(x \in R)\exists(n \in \mathbb{N})\ x < n$). When one uses smooth numbers, both these "pathologies" disappear. The price is, however, the weakening of arithmetic and logic.

[11] In Estrada-González (2014) cotopoi are called complement-topoi.

[12] The complement of an open set, representing negation in classical logic, is not open.

[13] One must be careful: performing the dualization is not mechanical. For instance, the co-exponential object, that must exist in cotopoi, is not related to implication, as it is the case with the exponential object in topoi, but to a connector called pseudo-difference.

the theory of partial inner product spaces is a good mathematical theory with interesting applications and the paraconsistent logic as its internal logic. This theory unifies some structures of functional analysis, such as generalized functions and scales on Hilbert or Banach spaces (Antoine, 1980; Antoine, Lambert, Trapani, 2011; Antoine, Trapani, 2009, 2010).

Let us then extend Bell's program beyond local frameworks and admissible transformations to any categories and functors between them. It is no longer a program but rather an idea, the aim of which is to organize certain intuitions related to the philosophy of mathematics. To the conglomerate of all categories and all functors between them we attach the label the *field of categories* (or *categorical field*). The label of *field* is intended to emphasize an undetermined character of this conglomerate (it is obviously not even any kind of "category of categories"). For some concrete purposes we might narrow it to certain of its subfields; for instance to $n$-categories, for some $n$, to coherent categories, to all topoi, etc.

The categorical field is not only a huge collection of categories, but also a highly dynamic entity with a lot of interactions given by functors between categories. To better emphasize this dynamical character of the categorical field, we could adopt the point of view of the one-type category theory. In this point of view, the categorical field is a field of "pure activity", consisting only of functors (with identity functors playing the role of categories [Heller, 2016]).

Let us notice that in the categorical field logic is a "local variable", in the sense that each category has its internal logic. To the field of categories there corresponds the "field of mathematical theories", and functors Lang and Syn are responsible for organizing the interaction between syntax and semantics of mathematical theories.

It goes without saying that the concept of the categorical field is far from being a well defined mathematical (or metamathematical) concept. And it is not intended to be. I think, it shows in the new light the nature of mathematics on its most global scale, and testifies to the capability of the category theory in disclosing deep structural aspects of mathematics.

## 5 Gödelian Limitations

Our next excursion into philosophical consequences of the categorical field idea is connected with Gödel's theorems. Putnam, in his paper "Mathematics without Foundations", quotes a not uncommon opinion according to which "Gödel's theorem suggests that the truth or falsity of some mathematical statements might be impossible in principle to ascertain", and remarks that "this has led some to wonder if we even know what we mean by »truth« and »falsity« in such a context" (Putnam, 1967). Indeed, one often hears about the crisis at the foundations of mathematics. Let us take a closer look at this problem. As it is well known, Gödel's theorems demonstrate the inherent limitations of every formal axiomatic system containing basic arithmetic (such as Peano axiomatic system). The first Gödel theorem states that no such axiomatic system can prove all truths about the arithmetic of natural numbers: there will always be true statements about natural numbers that cannot be proved within this axiomatic system. The second Gödel theorem demonstrates that no such axiomatic system can prove its own consistency.

These theorems are often interpreted as showing the collapse of the Hilbert program, the aim of which was to formulate a complete and fully consistent set of axioms for the whole of mathematics. This was supposed to be a remedy for the critical situation in the foundations of mathematics after recently discovered inconsistencies and paradoxes. After Gödel's theorem the crisis even deepened.

Gödel's achievement paved the way for further discoveries. Let us mention Tarski's theorem on the formal undefinability of truth, Church's theorem that Hilbert's *Entscheidungsproblem* is unsolvable, the Löwenheim–Skolem theorem, stating that if a theory has an infinite model, then it has an infinite number of other models, and Turing's theorem that there is no algorithm for solving the halting problem (for all these theorems see Smith [2007]).

If we agree to look at mathematics as at the field of categories, and take into account the fact that logic is a "local property" of the field (i.e. internal logic changes depending on category), then it becomes clear that the "Gödelian crisis" is also only a local effect. Or more prosaically: Gödel's theorems and their consequences are valid only in categories which contain basic arithmetic (for the sake of concreteness, let us think about Peano's arithmetic). Peano's arithmetic itself can be formulated as a type theory, and it can be shown that Peano's axioms hold in any topos, i.e. they are available in the internal language of any topos (Lambek, Scott, 1986, p. 145), and each topos, in which Peano's axioms hold, has an NNO (p. 189). However, there are many categories, having NNO, in which Peano's axioms do not hold (Awodey, 2010, p. 246).

The conclusion is that there are vast domains of mathematics, outside of the world of topoi, which are immune to Gödel-type limitations.[14] The Hilbert program cannot be revived not so much because of Gödel's theorems, but rather because of the nature of the categorical field, in which logic is but a "local variable".

## 6 Beyond The Formalism

My main concern in the previous sections was about formalised mathematical theories and what the theories are about (their semantics). In the present section, I want to introduce into the orbit of my interest physical theories. Their linguistic (syntactic) aspect does not differ much from that of mathematical theories (although physical theories are rather seldom fully formalised), and their categorical semantics could in principle be constructed. The essentially new aspect, when dealing with them, is the reference to the physical world.

In the philosophy of science, there are two main approaches to scientific theories: the syntactic approach and the semantic approach. Roughly speaking, the syntactic approach regards scientific theories more or less in the spirit of Definition 1, section 2. It emerged out of classical works of Carnap (1937), Hempel (1966) and other authors remaining under the influence of Logical Positivism. Scientific theories are sets of sentences in the language of a given science which should be subject to the logical analysis. This approach, once known as the "received view", met later with a strong criticism as being faraway from the scientific practice. According to its rival view, the so-called

[14] The above remarks inspire the following idea. As it is well known, any formalized axiomatic system can be organized as a category (with axioms and theorems as objects and deduction chains as morphisms). It would be interesting to investigate the internal logic of such categories, for instance, the internal logic of the category of the Peano axiomatic system.

semantic approach, propagated by Suppes (1960), van Fraassen (1989) and others, a scientific theory should be reconstructed in terms of the class of its models, and by taking into account its mathematical apparatus rather than predicate logic.

In the light of the analysis carried out in the preceding sections, it is clear that to base our approach to theories on an opposition between syntactic and semantic view is unjustified. As we have seen, in formal mathematical theories, syntax and semantics strongly interact with each other (via the Lang and Syn functors), and there is no reason to suspect that in not fully formalised mathematical and physical theories things are much different. Already Halvorson and Tsementzis (2017) have persuasively argued that a categorical approach to the problem of scientific theories "can transcend the syntax-semantics dichotomy in 20th century philosophy of science".

Let us notice, however, that having a formalized theory $T$, we – so to speak artificially – create the category $C$ providing the semantics for $T$. Loosely speaking, the theory $T$ is about what happens in $C$. Relations (1) and (2) describe interaction between syntax and semantics of $T$. Let now $T$ be a physical theory. Of course, we could in principle reconstruct its formal semantics, but doing physics in this way would be impractical, especially as far as more advanced physical theories are concerned. Usually, physicists think about their theories as being about a certain domain of physical reality. By using analogy with our previous analyses, let us try to reconstruct the interaction between the structure of a mathematized physical theory $T$ and its "natural semantics" $NS$, i.e. the domain of the physical world the theory $T$ investigates. It goes without saying that the following reconstruction is both informal and simplified. The process of reconstruction, that will be sketched here, is usually extended in time. The "we", appearing in this description, could denote several generations of investigators.

The process starts with the delineation of this domain or aspect of the world that will become an $NS$ for the future theory $T$ (in fact, this process is not completed until the theory $T$ is formulated; in what follows, we disregard the time factor).

The initial information about $NS$ consists of the so far existing knowledge and what can be termed "learned intuition". Basing on this, one tries to reconstruct an internal logic inherent in $NS$, by using various methods. The logic being reconstructed is implemented into the language of mathematics rather than into a logically informed language. The result of this – usually, long and laborious – process is a mathematized physical theory $T$. The process of its creation can schematically be presented (in a loose analogy with the functor Lang) as a "functor"

Mat: Natural Semantics → Physical Theories.

"Functor" (in quotes) because neither "Natural Semantics" nor "Physical Theories" is a category.

Having the theory $T$, we deduce from it some dependencies between measurable quantities (observables). We might say that when doing so we define (in a loose analogy to the functor Syn) a "functor"

Measur: Physical Theories → Natural Semantics.

We perform actual measurements in $NS$, that is we test the physical theory $T$, and obtain a new information about $NS$.

In this way, the theory $T$ represents, in a sense, its natural semantics $SN$. Obviously, between $T$ and $NS$ there is neither an identity, nor any kind of isomorphism, but rather some sort of adjointness (again in a loose analogy with the adjoint functors Lang and Syn). Owing to this adjointness, we are entitled to assume that what has been proved in $T$, has its counterpart in $NS$. This is how physical theories enrich our knowledge of the world.

## Appendix: Many Type Theory

Many type theory is given by

1. a collection $S = \{X,Y,\ldots\}$ of types;
2. a collection $F = \{f,g,\ldots\}$ of function symbols; each function symbol is given together with the types of its arguments and the type of its value;
3. a collection $R = \{U,W,\ldots\}$ of relation symbols; each relation symbol is given together with the types of its arguments
4. and possibly a collection of constants: $a,b,\ldots$, each constant together with its type.

It is also assumed that each type $X$ has infinitely many variables: $x_1, x_2, \ldots$ of this type. Then one builds up terms and formulae of the theory in the following way

- Each variable (or constant) of type $X$ is a term of this type.
- Suppose $t_1, \ldots, t_n$ are terms of types $X_1, \ldots, X_n$, and $f : X_1 \times \ldots \times X_n \to Y$ is a function symbol, then $f(t_1, \ldots, t_n)$ is a term of type $Y$.
- If $R \subseteq X_1 \times \ldots \times X_n$ is a relation symbol having arguments of types $X_1, \ldots, X_n$ and $t_1, \ldots, t_n$ are terms of types $X_1, \ldots, X_n$, then $R(t_1, \ldots, t_n)$ is an atomic formula.
- If $t$ and $t'$ are of the same type, then $t = t'$ is an atomic formula.
- The symbols $\top$ (true) and $\bot$ (false) are atomic formulae.

Having atomic formulae, one constructs more complicated formulae with the help of the connectives $\wedge$, $\vee$, $\Rightarrow$, $\neg$, and quantifiers $\forall$, $\exists$ in the standard way. For details see Mac Lane, Moerdijk (1992, pp. 527–530).

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