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Rozprawa

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ON SUPRASUBJECTIVE EXISTENCE IN MATHEMATICS

Summary: The professional mathematician is a Platonist with regard to the existence of mathematical entities, but, if pressed to tell what kind of existence they have, he hides behind a formalist approach. In order to take both attitudes into account in a possibly serious way, the concept of suprasubjective existence is proposed. It involves intersubjective existence, plus a stress on objectivity devoid of actual objects. The idea is illustrated, following William Byers, by the phenomenon of the rainbow: it is not an object but can be said to possess a subjective objectivity.

Key words: mathematics, Platonism, formalism, existence, objective, subjective, intersubjective, suprasubjective, culture, rainbow.

1 The Problem

The problem of the existence of numbers and other mathematical entities is among the most fundamental, the most studied and the most divisive in the philosophy of mathematics. Some opt for realism, often called Platonic, according to which numbers, circles, functions, spaces, etc. exist objectively in a separate realm. Others deny such non-physical existence, and opt for another extreme, that is, formalism, according to which only symbolic expressions exist, and mathematical objects are fictions. A milder position is possible: objects

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are mental constructions, and there exist various options regarding possible constructions. The conflicting views coexist, and the problem which of them is right seems undecidable. This is a deeply frustrating, even though not unfamiliar, situation for philosophers.

At the same time, from the perspective of working mathematicians, the problem seems virtually non-existent. Numbers and other items are treated simply as pre-existing objects. When asked if those objects really exist, mathematicians hesitate and often manifest the attitude which is aptly summarized by the classic saying that mathematicians are Platonists on weekdays and formalists on weekends (Hersh, 1979, p. 32). When doing mathematics, they see mathematical objects as real and objective; they perceive them as existing somewhere out there. When asked about the situation, and forced to philosophize, they lose certainty about the existence of numbers and other objects and hide behind a formalist position: mathematics is just a game, dealing with symbols. These contradictory views concerning existence in mathematics are manifested not just by greenhorns, but also by the most advanced and brightest professional mathematicians.

Usually philosophers see acceptance of two contradictory views concerning basic questions as a weakness, something that should be overcome. They believe that it would be desirable either to adopt one and argue in its favor, or to find a third position that would illuminate the other ones. Mathematicians apparently disagree, at least with regard to this issue: they are ready to support both realism and formalism. Maybe they would not express both in one utterance, but they are attracted to both and, *de facto*, do manifest the double position. This is very nicely expressed by Paul J. Cohen, an outstanding mathematician, who introduced an innovative method to prove the independence of the Continuum Hypothesis and the Axiom of Choice from the standard axioms of set theory. According to him:

> The Realist [Platonist] position is probably the one which most mathematicians would prefer to take. It is not until he becomes aware of some of the difficulties in set theory that he would even begin to question it. If these difficulties particularly upset him, he will rush to the shelter of Formalism, while his normal position will be somewhere between the two, trying to enjoy the best of two worlds (Cohen, 1971, p. 11).

Formalism is, of course, a relatively new position, although related to much older nominalist views. It spread when modern abstract mathematics developed and the existence of alternative geometries, and later of the wealth of other mutually incompatible mathematical theories, was made clear. In contrast, realism is a traditional view, traceable to Plato. One can argue (Król, 2015) that Platonism is necessary for mathematical activity, since it is the method of abstract mathematics as we have known it since the ancient Greeks. Indeed, Brouwer has convinced us that the very use of the law of excluded middle often reveals the Platonic conviction that a number preexists and that is why we may assume that it has or does not have a certain property.

Mathematicians seem to hold contradictory views about the nature of their subject matter. What should be the philosophers' reaction?

2 The Proposal

Mathematicians apparently accept both Platonic and formalist conceptions. My proposal is straightforward: Why don't we accept this reality in a most serious manner? If we agree that there is genuinely nothing more important to say philosophically about the existence of mathematical objects than what mathematicians express by their double position, what will follow concerning the problem of existence in mathematics? As a matter of fact, the wish to accept the double position, has been expressed in recent discussions on the philosophy of mathematics. It is implied by the move toward pluralism, extensively argued for by Michèle Friend (2013). It is also consciously advocated by William Byers (2017) and to some extent by some other authors of the volume honoring Reuben Hersh (Sriraman, 2017). For example, Hersh himself writes that "learning and teaching mathematics" is about "processes and concepts that are *taken for granted as real entities*" and at the same time they are not "real entities »out there« – independent of human apprehension" (2017a, p. 42).

I believe that the attempt to accept the ambiguity resulting from retaining both a realistic and a formalist approach is not just an intellectual exercise, but it is also a proper way of handling the problem. The attitude of mathematicians should be seen as an essential ingredient of the problem, not just a secondary circumstance. Rather than impose solutions, philosophers should take into account the mathematical experience. I mean the genuine experience, which is what the philosophers of mathematics have been trying to do in the last two decades or so (cf. Corfield, 2006). By the way, I believe that this experience is available not only to professional mathematicians, but also to philosophers and all those who try to do mathematics rather than just read about it.

Thus the idea is to retain the double positon. This is not an easy step. For example, Cohen himself decided to choose the formalist position (Cohen, 1971, p. 13). He did it following Abraham Robinson's lecture, *Formalism'64* (1965). At the same time, he admitted that there was a "great esthetic temptation [...] to accept set theory as an existing reality" (Cohen, 1971, p. 15).

The double position, the enjoyment of the best of two worlds, suggests that we need a view of numbers, sets and other mathematical entities that incorporates both the realist acceptance of their existence and the formalist denial thereof. Statements about mathematical objects can then be evaluated according to realism as well as according to formalism. In other words, with the same seriousness we should see mathematical truth as correspondence and as coherence.

One approach that seems, at first glance, to fulfill the double role can be called intersubjectivism. Intersubjective existence would mean existence beyond the subjectivity of an individual grasp of the world, but would not assume absolute existence completely devoid of a subjective perspective. Existence would be relative to communities of discourse. It would be a cultural creation, present between subjects, but not completely beyond human subjects. The book *What Is Mathematics, Really?* by Reuben Hersh, based on sound knowledge of living mathematics, proposing a "humanist philosophy of mathematics" (Hersh, 1997, p. 246), is a good example of this approach. According to it, mathematics is "a social-cultural-historical reality," so it is "»inner« with respect to society at large, »outer« with respect to you and me individually" (Hersh, 1997, p. 17). Other authors, like Wilder in (1981), offer even more explicitly cultural accounts of mathematics. Among the earlier of such approaches is the thesis by the anthropologist Leslie White: "mathematics in its entirety, its «truths» and its «realities,» is a part of human *culture,* nothing more" (White, 1947/2006, p. 307). According to him, mathematics has "the sort of reality possessed by a code of etiquette, traffic regulations, the rules of baseball, the English language or rules of grammar" (White, 1947/2006, p. 319).

And this is the point at which the cultural interpretation is unsatisfactory. Though mathematics is a cultural construction, it is not arbitrary. We have to adjust to its exigencies in a way different than when we adopt the rules of grammar, let alone baseball. Mathematics seems to be objective, or independent of our needs, decisions and actions, in a stronger, more fundamental way. When we enter a framework, a mathematical theory or a realm of mathematical entities, we feel ourselves to be subjected to rigid constraints which no cultural effort can overcome. For example, the move from real numbers to complex numbers is considered by mathematicians not as just one possible culturally defined extension of the reals, but as the right one, imposed by the subject matter. Indeed, we feel we have discovered the truth of that matter – a timeless truth.

Therefore, even though the intersubjective interpretation of mathematical existence is philosophically attractive, it is not sufficient. It does not involve as much objectivity as it should. The realm of mathematics seems objective to mathematicians. When we try to find solutions to problems, we need to painstakingly search through the reality we find ourselves in. That reality seems as objective as anything in the real world. It is independent of us, not only us as individuals, but also us as human culture. It is beyond subjects. Put a bit differently, there exist mathematical facts. And they cannot be explained as indirect consequences of our cultural assumptions, that is, no such explanation will be satisfying to mathematicians. A dimension of genuine objectivity must necessarily be involved in any satisfactory account. The simplest examples are provided by natural numbers. Given any collection of prime numbers we can indicate a larger prime number, as was demonstrated by Euclid. This is an objective fact. Further, either there are finitely many pairs of primes or not, but we still do not know which is the case. One or the other is a fact, and we have no influence upon it. This statement is obvious to every mathematician.

Objective facts of this sort, completely independent of any subject, induce realistic views, that is, the conviction that numbers are (Platonic) objects. Yet to admit the existence of such fairy tale objects seems unwarranted, and as hard to accept by a serious adult as are distant ridiculous mythologies. That is why the formalist approach is attractive.

A way out of the contradiction between objectivist realism and subjectivist formalism, the philosophical third position, would be to accept objectivity and reject objects. Then both of those positions can be maintained. And they would be adopted in a manner natural for a mathematician. Thus, whereas there are objective facts of the matter, there is no correspondence between our concepts and some independently existing objects. In the well-known earlier book (1981), written with Philip J. Davis, Hersh expressed this sentiment:

> Do we really have to choose between a formalism that is falsified by our everyday experience, and a Platonism that postulates a mythical fairyland where the uncountable and the inaccessible lie waiting to be observed by the mathematician whom God blessed with a good enough intuition? [quoted from the later edition, 1995, p. 448].

This remark is directed against the modern followers of Platonism, like the leading logician Kurt Gödel, who famously wrote about sets that

> the assumption of such objects is quite as legitimate as the assumption of physical bodies and there is quite as much reason to believe in their existence. They are in the same sense necessary to obtain a satisfactory system of mathematics as physical bodies are necessary for a satisfactory theory of our sense perceptions (1944/1990, p. 128).

The idea that objectivity can occur in the absence of objects has been expressed by some philosophers of mathematics, e.g., William Byers (see Section 3). Interestingly enough, Gödel himself also said things to this effect. In his lecture (1951/1995, p. 311), he observed that “mathematical objects and facts (or at least *something* in them) exist objectively and independently of our mental acts and decisions,” which would be a restatement of Platonism if not for the mitigating phrase in the parentheses. A much stronger indication of the separation between objectivism and objects is reported by Hao Wang. According to it, Gödel emphasized “(1) the fallibility of our knowledge; (2) the epistemological priority of objectivity over objects.” (Wang, 1996, p. 210). And later he added, with regard to sets, that “the question of the objective existence of the objects of mathematical intuition [...] is not decisive” (Gödel, 1964/1990, p. 268).

Having separated objectivism from objects, we enter rather unexplored ground. Where this objectivity comes from, if not from objects, seems to be left unexplained. We can try to speculate that this restricted form of objectivity is grounded in the physical or the biological or the conceptual world, that it is caused by our perceptual or knowledge-acquiring apparatus, by our biological or social composition, but I am not trying to argue here for any specific explanation. Maybe other dimensions of broadly conceived reality bring the feeling of objectivity (without objects). I think that we need to move away from objects and, as a result, accept the associated uncertainty. Perhaps this approach was indicated by Hersh's general remark that "mathematics is part of human culture and history, which are rooted in our biological nature and our physical and biological surroundings" (Hersh, 1997, p. 17).

Anyway, we are led to an extension of intersubjectivism. In mathematics, and perhaps elsewhere as well, there is something beyond or above the subjective, as well as above the intersubjective. I believe this approach deserves a name. Because of the addition of something above the intersubjective I propose to call it *suprasubjective*. The schematic formulation would read:

Suprasubjective = intersubjective + objective without objects.

3 A Metaphor

William Byers, who is both a mathematician and philosopher of mathematics, offers deep and novel insights into real mathematics. In his remarkable book (2007) demonstrating the importance of paradox, ambiguity, and even contradiction for living mathematics, he introduced the idea of the objective subjectivity or subjective objectivity of mathematics. Mathematical patterns, he says, "contain both objective and subjective perspectives" (Byers, 2007, p. 345). I believe he is pointing in the same direction as the one that is being sketched here. Byers quotes George Lakoff and Mark Johnson who in their classic book (1980) on metaphors propose to go beyond "the myths of objectivism and subjectivism." Yet the most illuminating metaphor is borrowed by Byers from Nick Herbert, who in his book (1985) on quantum mechanics uses the phenomenon of the rainbow to illustrate the fact that the quantum world is objective, but it is not made of objects; it is objectless. Similarly, a rainbow is an objective phenomenon, but it is not a "thing."

The rainbow has been fully explained by science. Yet it retains not only its old power and beauty but also possesses the rare quality of being both fully objective and completely viewer dependent. It is objective because it can be photographed and it is intersubjective as everybody sees it the same way if put in the same place. At the same time, changing the place changes slightly the location of the rainbow. It seems to be there, but there is literally nothing there where it seems to emerge. It is not a physical object, but it is a physical phenomenon. It exists for us, but it is not just a hallucination. It is also much more concrete, stable, and repeatable than a mirage in the desert. The rainbow is indeed a perfect example of a mixture of objective and subjective existence. One cannot be separated from the other. A rainbow is not an object, but it is objective subjectively or subjective objectively.

Byers suggests that mathematics can be seen in a similar way: "when one encounters some mathematical entity it is never devoid of a point of view" (Byers, 2007, p. 345). And yet it remains objective. And it does not need to be an object. It is like a rainbow and the quantum world. If we agree that mathematical entities are of this character, we get more than subjective and more than intersubjective existence. We also get objectivity without objects. In one word, suprasubjective existence.

The rainbow metaphor is also reassuring in another way. We know the optical mechanism behind the phenomenon. However, even if we did not, as was the case for centuries, we could still see the rainbow as an example of subjective objectivity. There would remain a mystery regarding the mechanism, but, according to our scientific approach, we would be sure that one day the nature of the phenomenon would be discovered. I think that much the same is true about mathematics. We have entered unknown ground, we do not know how the suprasubjectivity arises, what kind of mechanism makes it work, but we can hope that one day the matter will be explained.

4 An Assessment

As mentioned above, the proposal made in this paper is implicitly contained in suggestions made by other authors. Hersh uses the term "object" for a cultural creation: "Mathematical entities are real objects and they are part of culture," that is, they are "sociocultural entities and intersubjective" (Hersh, 2017b, p. 361). Hersh's example, the US Supreme Court, which is not physical or merely mental, and does influence our lives in a very real way, is illuminating, but also misleading. This "object" is largely a social convention, while mathematics seems to be much more independent of any conventions. I believe that it is better not to use the term "object" but to retain "objectivity" and use the term "suprasubjective".

Something akin to it has been recently clearly expressed by Byers. He says, "we feel that that mathematics *is* at arms' length and that it is objective and timeless" (Byers, 2017, p. 46). Then he distinguishes strong from weak objectivity. Mathematics, he claims, "is objective in the weak sense but not in the strong – free from prejudice and arbitrary opinion but not independent of intelligence" (Byers, 2017, p. 48; see also Byers, 2015). I believe that the concept of suprasubjectivity is helpful here: we reassert intersubjectivity and indicate restricted objectivity while avoiding talking about objects.

Perhaps the approach adopted in this paper constitutes an interpretation of, or a development within, the Popperian idea of World 3. According to it, as summarized by Eduard Glas,

> mathematical objects, relations and problems can be said in a way to exist independently of human consciousness *although* they are products of human (especially linguistic) practices. […] Once created, however, this product assumes a partially autonomous and timeless status […], that is, it comes to possess its own objective, partly unintended and unexpected properties, irrespective of when, if ever, humans become aware of them (Glas, 2005, p. 292).

I believe that this objectivity needs to be emphasized in a more fundamental and explicit way than is usually done when this intuition is expressed. That is why we need a term like "suprasubjective."

I believe that the concept of suprasubjective existence can be partially acceptable to the main schools offering answers to the problem of existence in mathematics. This new concept does take into account the main insights of the competing schools. Thus, first, it should be sufficiently satisfactory for realists since it involves objectivism, allows for the presence of the rigidity and mind-independence that is characteristic of mathematical constructions, even those freshly invented by us. The realists would oppose, however, the rejection of independently existing objects.

The concept of suprasubjective existence should be sufficiently acceptable for formalists as it assumes the crucial role of our standpoints, including the language and formulations. They, or rather the extreme formalists (if there are any), would, however, be dissatisfied with the approach that there is something more than meaningless games, whose choice is guided by extra-mathematical needs, be they esthetic or pragmatic.

The proposal should be satisfactory for constructivists and culturalists since it involves human activity and the intersubjective dimension as its main pillar. It adds, however, the most serious objectivity claim. At the same time, it rejects Platonic objects, leaving open more indirect ways of justifying objectivity. The understanding of those ways may require a hitherto unknown development of our descriptions of interactions between human subjects, their ideas and their physical, biological, and social environment.

And, above all, the concept of suprasubjective existence of mathematical entities should be highly satisfactory for mathematicians. It harmonizes, I believe, with their instincts.

The present proposal has arisen from an attempt to listen to mathematicians rather than impose interpretations on them. As Michèle Friend explains the concept of pluralism in philosophy of mathematics, it is based on "unprejudiced observation of mathematical practice and a desire to encompass and accommodate as wide a variety of practices as is coherently possible" (Friend, 2013, p. 257). Whether this is a sound way of doing the philosophy of mathematics is bound to remain debatable.

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