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Article

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THE NATURE OF PROPOSITIONAL DEDUCTION— A PIAGETIAN PERSPECTIVE

SUMMARY: Logic was once thought to describe the laws of thought; however, a plurality of logics has now replaced classical logic, obscuring rather than clarifying the nature of deduction with an embarrassment of riches. In cognitive science, on the other hand, logic is not thought to be a psychological theory of human reasoning. Research on human reasoning has traditionally focussed on deduction, although human reasoning is thought to be much richer, and two competing theories dominated discourse in cognitive science—the syntactic, formal-rule, and the semantic, mental-model theory. Jean Piaget also proposed a psychological theory of reasoning, but, in contrast to these classical theories, he advocated an operatory theory. Deduction is an integral part of Piaget's theory, and, in this paper, I briefly outline Piaget's operatory theory of propositional reasoning before explicating the nature of deduction embodied in it. I conclude that the nature of propositional deduction according to Piaget lies in the interpropositional grouping, a natural structure at the heart of propositional reasoning constituted by a closed system of operations of thought regulated by laws of transformation. I then argue that the nature of propositional deduction lies specifically in the lattice constituted by the inclusion/order relations between the propositions of the interpropositional grouping. Piaget did not conceive of the interpropositional grouping as a logic; nevertheless, I wind up arguing that a logic conceived as Piaget intimated would complement the plurality of logics with a natural logic.

KEYWORDS: operations of thought, grouping, structure, propositional reasoning, propositional deduction, Boolean algebra, lattice, logic.

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1. Introduction

From a logical perspective, an inference can be analysed into input, premisses, and output, conclusions, and a rule of inference governing the transition from premisses to conclusions. Inferences are then valid if the transition from premisses to conclusions occurs according to the rules of inference, and deductive inferences, considered paradigmatic of rational thought, are those whose conclusions are necessarily true if the premisses are true. Whilst the characterisation of inferences may not be controversial, what logic is beyond the study of inferences is (Hintikka, Sandu, 2007, Section 1). It is widely accepted today that logic, once synonymous with classical logic, has branched into a plurality of approaches, characterisations, and often rival concepts of validity (Restall, Beall, 2000; 2001; Russell, 2019). When asked what logic is, modern logicians, in contrast to their pre-20th-century colleagues, thus flush with an "embarrassment of riches" (Hintikka, Sandu, 2007, p. 13; Jacquette, 2007, p. 3). Equally, philosophers, logicians, and psychologists who seek the nature of deduction in logic alone (George, 1997; Posy, 1997) are chagrined.

Highlighting a mismatch between classical logic and human reasoning, Johnson-Laird concludes that "[l]ogic is an essential tool for all sciences, but it is not a psychological theory of reasoning" (Johnson-Laird, 2006, p. 17). Moreover, human reasoning is not thought to be synonymous with deduction (Harman, 1984; 1986; van Benthem, 2007; 2008); nevertheless, psychological research has tended to focus on deductive inferences (e.g., Johnson-Laird, 2006, p. 3). Broadly, cognitive scientists entertained three psychological theories of deductive reasoning: deduction is either a process based on factual knowledge; a syntactic, formal process, or a semantic process based on mental models (Johnson-Laird, 1999). Since the knowledge-based theory of deductive reasoning relies on memory of prior inferences, it is unable to account for inferences that are confidently made without precedents or even prior knowledge of the subject matter involved. Apart from tailor-made theories for particular aspects of reasoning, discourse on human reasoning was therefore portrayed as a two-horse race between the syntactic, formal-rules and semantic, mental-model theories (e.g., Byrne, Johnson-Laird, 2009; Johnson-Laird, 1999; 2006, Chapter Introduction; Rips, 2008).

In essence, advocates of formal-rule theories maintained that reasoners recognise logical forms in premisses and apply rules of inference akin to logical rules when inferring. Clearly, logic is the source of inspiration for these theories, and, rather than being an embarrassment, the plurality of logics could serve as a rich source of hypotheses for the formal rules of inference employed in reasoning (e.g., Stenning, van Lambalgen, 2008; 2011). Advocates of the mental-model theory, on the other hand, maintained that reasoning is a process of envisaging possibilities. In essence, reasoners construct models of possibilities consistent with the premisses given and draw conclusions based on them. In contrast to the syntactic, formal-rules theories and classical logic, content and context rather than form therefore play an important role in the mental-model theory. However, portraying the discourse as a two-horse race between rival psychological theories of reasoning was misleading. Jean Piaget also proposed a psychological theory of human reasoning, and, being based on operations of thought, it is fundamentally different from both the formal-rules and mental-model theories.

Piaget's theory has all but disappeared from mainstream debate on reasoning. A reason for its disappearance may lie in Piaget's theory being classified as an outdated progenitor of formal-rule theories (e.g., Johnson-Laird, 1999, p. 114; 2006, p. 14); "reasoning is nothing more than the propositional calculus itself" (Inhelder, Piaget, 1958, p. 305; Johnson-Laird, Byrne, Schaeken, 1992, p. 418), for example, is the citation Johnson-Laird uses to support a formal-rule-theory interpretation of Piaget's theory. However, Johnson-Laird does concede that "Piaget's views on logic are idiosyncratic" (Johnson-Laird, Byrne, Schaeken, 1992, p. 418), and "[i]t is not always easy to understand Piaget's theory" (Johnson-Laird, 2006, p. 249). Johnson-Laird's confessions express popular assessments of Piaget's theory of reasoning among Anglophone cognitive psychologists (Bond, 1978; 2005), and they corroborate Piaget's own impression that his work was poorly understood (Smith, Mueller, Carpendale, 2009, pp. 1–10).

Difficulties in understanding Piaget's theory are exacerbated by the inaccessibility of his original works in a predominately Anglophone research environment. He wrote in French, and translations into English are selective and not rarely dubious in quality (Smith, Mueller, Carpendale, 2009, pp. 28–44). Returning to the citation Johnson-Laird uses to substantiate his claim, Lesley Smith considers "reasoning is nothing more than the calculus embodied in propositional operations" (Smith, 1987, p. 344) to be a more faithful rendition of the French original. The difference in translations may seem minimal, but this paper shows that the operatory standpoint is essential for the correct understanding of Piaget's theory of human propositional reasoning and the nature of propositional deduction in particular.[1]

I begin my exposition of the operatory nature of deduction by introducing operations of thought (Section 2). Piaget attempted to cast the calculus embodied in propositional operations in a formal language by using the algebraic tools logic put at his disposal, and, due to its formal appearance, the calculus might easily be mistaken for a logic. Before setting out the interpropositional grouping, I therefore briefly explicate "psycho-logic" (Section 3) to clarify Piaget's intentions. I then go on to set out the calculus embodied in propositional operations of thought (Section 4), beginning with the most elementary interpropositional grouping constituted by the affirmation and negation of a single proposition (4.1) before setting out its systematic extensions to multiple propositions. At this point, I would like to apologise to the reader for rehearsing Piaget's operatory theory of reasoning in such detail, especially to those familiar with his work.

[1] For the sake of brevity, the additional attributes are assumed on writing "reasoning" and "deduction" from now on.

In view of misconceptions surrounding Piaget's theory, however, I feel obliged to adumbrate and justify my interpretation.[2] The extensions are based on implication, and I set out the four distinct forms of implication inherent in the interpropositional grouping (Sections 4.2 and 4.3), before presenting Piaget's analysis of these forms from the point of view of deduction (Section 5). I then characterise the nature of deduction according to Piaget—broadly first, from a diachronic then a synchronic perspective; in detail second, by focusing specifically on the forms of implication (Section 6)—before concluding (Section 7) with a brief remark on a ramification of the nature of deduction according to Piaget for logical pluralism.

2. Operations

By joining propositions to others using propositional connectives, such as "and" ($\wedge$), "or" ($\vee$), "if-then" ($\supset$), etc., compound propositions are constructed. The meaning of the compound proposition is then constituted by the meanings of the constituent propositions and the propositional connective involved. Just as compound propositions are composed of parts, the propositions themselves are also composite in nature. In contrast to compound propositions, however, the constituent parts are not propositions; "Mammals are vertebrates", for example, has a subject "mammals", predicate "vertebrates" and a logical constant "is". These parts can be substituted for others, and the meaning of the whole proposition is constituted by the meanings of its parts. Operations are intellectual activities that compose and decompose such connections between propositions or between the parts of propositions (Piaget, Grize, 1972, p. 9). Piaget denotes the former "interpropositional operations" and the latter "intrapropositional operations" (Piaget, Grize, 1972, pp. 34–35). In this paper, I focus on interpropositional operations although deduction also occurs in intrapropositional reasoning.

Whether intra- or interpropositional, Piaget describes the psychological nature of intellectual operations as follows:

> [O]perations are actions which are internalizable, reversible, and coordinated into systems characterized by laws which apply to the system as a whole. They are actions, since they are carried out on objects before being performed on symbols. They are internalizable, since they can also be carried out in thought without los-

[2] Partly due to no standard edition of Piaget's work in French existing and reliable English translations being few and far between, misconceptions of Piaget's work are abound. With readers' convenience in mind, some exegetical overlap is therefore inevitable (e.g., Winstanley, 2021). However, the manuscripts pursue entirely different questions on the basis of the exegeses: using Piaget's theory of reasoning as an example of how psychology may legitimately serve as logical evidence for logical theory, Winstanley (2021) focuses on the interface between logic and psychology and elaborates ramifications for anti-exceptionalism about logic; the current paper, in contrast, elucidates the nature of propositional deduction according to Piaget and therefore has a psychological focus.

> ing their original character of actions. They are reversible as against simple actions which are irreversible. In this way, the operation of combining can be inverted immediately into the operation of dissociating, whereas the act of writing from left to right cannot be inverted to one of writing from right to left without a new habit being acquired differing from the first. Finally, since operations do not exist in isolation they are connected in the form of structured wholes. (Piaget, 1957, p. 8; see also Piaget, 1971a, pp. 21–22; 2001, Chapter 2; Piaget, Beth, 1966, p. 172; Piaget, Grize, 1972, p. 55)

Piaget used logical tools to represent the structured wholes formed by operations. However, precisely because of their formal appearance, it is important to nip misconceptions in the bud by clarifying how Piaget intended these models to be understood.

3. Psycho-Logic

Logic is concerned with what conclusions follow from what premisses, and it develops techniques for determining the validity of inferences. Piaget's theory, on the other hand, is not primarily concerned with logical consequence, and it does not provide techniques for assessing the validity of arguments (Grize, 2013). Piaget understood his theory in analogy to mathematical physics. Physics investigates the physical world experimentally, and its criterium for truth is correspondence with empirical facts; mathematics, on the other hand, is neither based on experiment nor does its truth depend on agreement with empirical facts. It is a formal science whose truth depends solely on the formal consistency of the deductive systems constructed. Drawing on both deductive and empirical sources, mathematical physicists, aiming to understand the physical world, apply mathematics to physics to construct deductive theories based on the experimental findings of physics. Like mathematical physics, Piaget (1957; see also Bond 1978; 2005) also envisages "psycho-logic" or "logico-psychology" as a *tertium quid*. On the one hand, psychology investigates mental life empirically, and its criterion for truth is correspondence with experimental facts; on the other hand, logic, like mathematics, is a deductive science concerned with formal rigour rather than correspondence with facts, and it has developed algebraic techniques. Psycho-logic is an application of the algebraic tools of logic to the findings of experimental psychology, and it aims to construct a formal theory based on the experimental facts of psychology. In other words, psycho-logic uses logic to model the structured wholes systems of operations form.

In the next section, the most elementary model of interpropositional operations is set out first, and it is followed by progressive extensions.

4. The Interpropositional Grouping

Piaget modelled the structured wholes operations of thought form with a grouping. Roughly, a grouping is a structure incorporating the reversible operations of its namesake, mathematical groups, and the non-reversible operations of lattices.[3]

4.1. A Proposition and its Grouping

Given a single proposition p and its negation $\bar{p}$, Piaget and Grize (1972, pp. 321–322) set out the operations of the grouping as follows:

(i) The direct operation unites p disjunctively ($\vee p$) with other propositions of the system. Since $\bar{p}$ is currently the only other proposition, $\bar{p} \vee p = T$, for example; T is, therefore, a compound proposition comprised of p and $\bar{p}$, and it is also part of the system.

(ii) The inverse operation unites the negation of p conjunctively ($\wedge \bar{p}$) with other propositions of the system; for example, $T \wedge \bar{p} = \bar{p}$, $p \wedge \bar{p} = o$, etc. o is therefore also a proposition of the system.

(iii) The general identity operation, denoted ($\vee o$), is a) the product of direct and inverse operations, $p \wedge \bar{p} = o$; and b) it leaves the propositions it is composed with unaltered; for example, $p \vee o = p$, $\bar{p} \vee o = \bar{p}$, $T \vee o = T$, $o \vee o = o$, etc.

(iv) Despite not being composed of direct and inverse operations, some operations also leave the propositions they are composed with unaltered much like the general identity; for example, $p \vee p = p$; $\bar{p} \vee \bar{p} = \bar{p}$; $\bar{p} \wedge \bar{p} = \bar{p}$; etc.

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[3] Mathematically, a group is a set of elements with a binary operation that combines any two elements of the set to form a third in such a way that the group axioms— associativity, identity, and invertibility—are satisfied.

A lattice, on the other hand, can be defined in two different ways. On the one hand, it is a partially ordered set in which any two elements have both a least upper bound (meet) and a greatest lower bound (join). A partially ordered set, poset $\mathcal{P}$ for short, is an algebraic system in which a binary relation $x \geqq y$ is defined satisfying the following postulates:

P_1:	For all x, $x \geqq x$	(reflexive property)
P_2:	$x \geqq y \wedge y \geqq x, x = y$	(antisymmetric property)
P_3:	$x \geqq y \wedge y \geqq z, x \geqq z$	(transitive property)

The binary relation satisfying these postulates is called an inclusion or order relation (Rutherford, 1966, p. 1). For elements x and y of a lattice $\mathcal{L}$, the meet is denoted $x \cup y$ and the join $x \cap y$.

Alternatively, a lattice is a set $\mathcal{L}$ of elements with two binary operations $\cap$ and $\cup$ satisfying the following postulates for all $x, y, z, \ldots$ of $\mathcal{L}$ (Rutherford, 1966, pp. 4–5):

$L_{1\cap}$: $x \cap y = y \cap x$	$L_{1\cup}$: $x \cup y = y \cup x$	(Commutative Laws)
$L_{2\cap}$: $x \cap (x \cap z) = (x \cap y) \cap z$	$L_{2\cup}$: $x \cup (y \cup z) = (x \cup y) \cup z$	(Associative Laws)
$L_{3\cap}$: $x \cap (x \cup y) = x$	$L_{3\cup}$: $x \cup (x \cap y) = x$	(Absorptive Laws)

Via the identity $y = x \cap y \equiv x \geqq y \equiv x \cup y = x$, the two definitions can be shown to be equivalent (Rutherford, 1966, Section 4).

(self-inclusions), and $\bar{p} \vee T = T$, i.e., $p \vee (p \vee \bar{p}) = (p \vee \bar{p})$ (absorptions). These operations are *special identities*.

(v) Finally, the compositions are only partially associative; e.g., $p \vee (\bar{p} \vee o) = (p \vee \bar{p}) \vee o = T$, whereas $p \vee (p \wedge \bar{p}) \neq (p \vee p) \wedge \bar{p}$ because $p \vee (p \wedge \bar{p}) = p \vee o = p$ and $(p \vee p) \wedge \bar{p} = p \wedge \bar{p} = o$.[4]

The first three operations are reversible like the operations of a group. Moreover, $T = p \vee \bar{p}$, $p \wedge \bar{p} = o$, as well as $p \vee o = p$ and $\bar{p} \vee o = p$; the group-like operations are therefore reminiscent of the laws of thought, excluded middle, non-contradiction, and the law of identity, respectively. The fourth operation, on the other hand, is characteristic of a lattice, and $p \vee p$ (self-inclusion) and $p \vee o$ (direct operation and the general identity), especially, limit the associativity characteristic of the operations of a group.

Since the direct operation operates on all the propositions of the system, it also combines $\bar{p}$ disjunctively with other propositions of the system so that $p \vee \bar{p} = T$, for example, and the corresponding inverse operation is $T \wedge \bar{\bar{p}} = T \wedge p = p$. Since the inverse operation $\overline{(\vee \bar{p})} = \wedge \bar{\bar{p}} = \wedge p$, conversely $\overline{(\wedge p)} = \overline{(\wedge \bar{\bar{p}})} = \overline{\overline{(\vee \bar{p})}} = (\vee \bar{p})$, Piaget and Grize (1972, pp. 321–322) define $\wedge p$ and $\vee \bar{p}$ as another reversible pair of operations in the grouping just like $\vee p$ and $\wedge \bar{p}$. These operations introduce their own special identities; for example, $p \wedge p = p$ (self-inclusion), but $p \wedge T = p$ and $\bar{p} \wedge T = \bar{p}$ instead of absorption. $\wedge T$ is the most general of these special identities, and, like the general identity operation, it leaves the propositions it is composed with unaltered; unlike the general identity, however, it is not composed of direct and inverse operations.

4.2. The Forms of Implication

Implication is one of the few logical operators already present in the elementary grouping involving the affirmation and negation of a single proposition (Piaget, Grize, 1972, p. 323),[5] and, by differentiating the implication $p \supset T$ into a chain of implications, the elementary grouping can be extended to multiple propositions as follows:

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[4] Another possible source of confusion needs to be nipped in the bud. The symbolism is familiar from propositional logic; however, it does not have the conventional meaning (Apostel, 1982). Piaget (Piaget, Beth, 1966, pp. 180–181) simply found it convenient to adopt the symbolism of propositional logic and give it a whole new meaning in the context of his operatory theory.

[5] $(T \wedge p) \vee (T \wedge \bar{p})$ is a composition of the operations of the elementary grouping. By substituting the p and T of the elementary grouping for p and q in column 7 of Table 1, the disjunctive normal form of the conditional $p \supset T = (T \wedge p) \vee (T \wedge \bar{p}) \vee (\bar{T} \wedge \bar{p})$ is obtained. However, it reduces to $p \supset T = (T \wedge p) \vee (T \wedge \bar{p})$ in the elementary grouping since $\bar{T} = o$ therefore $(\bar{T} \wedge \bar{p}) = o$.

$p \supset q; q \supset r; r \supset s; s \supset t \ldots u \supset T$

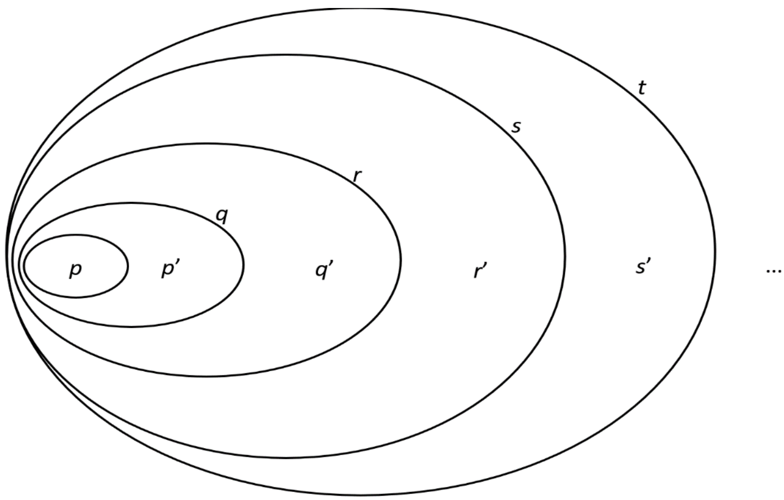
where q now plays the same role for p as T did for p in the elementary grouping; r, for q; s, for r, etc. By systematically elaborating the operations of the grouping on this chain of implications, Piaget and Grize (1972, pp. 324–325) discerned four distinct forms of implication.

4.2.1. Form I.

In Form I (Piaget, Grize, 1972, pp. 324–327), $q = pq \vee \bar{p}q$ expresses the common and non-common parts of p and q in analogy with $T = (T \wedge p) \vee (T \wedge \bar{p})$. The non-common part, $\bar{p}q$, is the relative complement of p in q, and Piaget denotes it p'; q can therefore be expressed more concisely as $q = p \vee p$'. Proceeding analogously for the other propositions in the chain, we have $r = q \vee q$', where $q' = r \wedge \bar{q}$; $s = r \vee r'$, where $r' = s \wedge \bar{r}$; etc. (see Figure 1).

Figure 1

Grouping of Implications—Form I



Note. Piaget calls $p, q, r, s, t, \ldots$ primary propositions and their relative complements $p' = q \wedge \bar{p}, q' = r \wedge \bar{q}, r' = s \wedge \bar{r}, \ldots$ secondary propositions. Primary propositions in the hierarchy are composed of the primary and secondary propositions of the previous level as follows: $q = p \vee p', r = q \vee q', s = r \vee r', \ldots$ (Piaget, Grize, 1972, p. 324, Fig. 46).

Using $\vee p$ and $\wedge \bar{p}$, one of the reversible pairs of operations of the elementary grouping (see Section 4.1), Piaget shows that Form I also constitutes a grouping:

(i) The direct operation $\vee p$ composes a proposition p with another proposition of the system to form an equivalence; e.g., $p \vee p' = q$; $q \vee q' = r$; etc.

(ii) The inverse operation $\wedge \bar{p}$ composes the negation of a proposition conjunctively with another proposition of the system; e.g., $q \wedge \bar{p} = p'$; $q \wedge \bar{p}' = p$; $\bar{p} \wedge \bar{p}' = \bar{q}$; $r \wedge \bar{p} = p' \vee q'$; etc.[6]

(iii) The general identity operation $\vee o$ is the product of the direct and inverse operations, e.g., $p \wedge \bar{p} = o$. Composed with other operations, the general identity leaves them unchanged; e.g., $p \vee o = p$; $\bar{p} \vee o = \bar{p}$, etc.

(iv) The special identities are self-inclusions; e.g., $p \vee p = p$, $\bar{p} \vee \bar{p} = \bar{p}$, $\bar{p} \wedge \bar{p} = \bar{p}$, etc.; and absorptions; e.g., $p \vee q = q$.[7]

(v) Associativity is limited because of the special identities.

As well as being a multipropositional differentiation of the implication present in the most elementary grouping involving the affirmation and negation of a single proposition, Form I thus also constitutes a grouping with $\vee p$ and $\wedge \bar{p}$ as direct and inverse operations (Piaget, Grize, 1972, pp. 324–325). Moreover, it is analogous to the inclusion of classes $P \subset Q \subset R \subset S \subset$, etc., familiar from biological taxonomies, genealogies, etc. Piaget (Piaget, Grize, 1972, p. 103) call *S*, *P*, *Q*, *R*, *S*, etc. *primary* classes, and these primary classes have relative complements *P*', *Q*', *R*' etc., which he calls *secondary* classes. The grandchildren of a grandparent *Q*, for example, are comprised of the children of one of *Q*'s children *P*, and their first cousins *P*'. In terms of primary and secondary classes, the classes constituting the nesting inclusions are therefore as follows:

$$P \cup P' = Q,\ Q \cup Q' = R,\ R \cup R' = S, \text{ etc.}$$

Let propositions *p*, *q*, *r*, *s*, etc. express the membership of an element *x* in the primary classes *P*, *Q*, *R*, *S*, etc. and *p*', *q*', *r*', etc., its membership in the secondary classes *P*', *Q*', *R*', etc. Clearly, if *q* is true, *x* is a member of $Q = P \cup P'$ therefore *x* is a member of either *P* or *P*', i.e., $p \vee p'$; Form I, therefore, corresponds to the nesting inclusions of classes typically found in Porphyrian trees. In fact, the intrapropositional operations on such classes also constitute groupings

[6] a) Unlike classical logic, negation of a single proposition is not a unary operator; it is equivalent to an inverse operation and therefore a binary operator; $\bar{p}$, for example, is equivalent to $\wedge \bar{p}$, i.e., $\bar{p} = T \wedge \bar{p}$, the relative complement with respect to *T*. $\bar{\bar{p}} = p$ is therefore equivalent to $\overline{(\wedge \bar{p})} = p$, the complement of the complement of *p* with respect to *T*, rather than $\bar{p} \wedge \bar{p} = \bar{p}$.

b) Moving a proposition from one side of an equivalence to the other is equivalent to applying the inverse operation; e.g., if $p \vee p' = q$, then $(p \vee p') \wedge \bar{p}' = q \wedge \bar{p}'$, i.e., $p = q \wedge \bar{p}'$; similarly, $p' = q \wedge \bar{p}$; $(p \vee p') \wedge \bar{q} = o$, etc.

[7] Due to the special identities, rules of composition must also be observed when propositions are transferred across equivalences; e.g., $(p \vee p) = p$ cannot become $p = (p \wedge \bar{p})$ when transferring $\vee p$ from the left to $\wedge \bar{p}$ on the right since $(p \wedge \bar{p}) = o$ and $p \neq o$.

(Piaget, Grize, 1972, Chapter II), and Form I models one of these groupings in terms of propositions (Piaget, Grize 1972, p. 324). For the present purposes, however, the correspondence with nesting inclusions of classes simply facilitates the recognition of valid inferences. Clearly, if x is a member of a class, it is automatically a member of all of its superclasses; primary p, q, r, etc. and secondary p', q', r', etc. propositions, therefore, imply primary propositions of higher rank; e.g., $p' \supset t$ or $r \supset u$, etc. Conversely, if x is a member of a primary class, it must be a member of one of the disjoint classes composing it; each primary proposition therefore implies those propositions composing it but as a whole; e.g., $s \supset (p \vee p' \vee r')$. Finally, any subclass of a primary class of higher order can be determined by eliminating relative complements; any proposition can therefore be inferred from those of higher rank by negating complementaries; e.g., $q' = t \wedge \bar{s}' \wedge \bar{r}' \wedge \bar{q}$ (Piaget, Grize, 1972, p. 326). As well as the membership of elements in classes, propositions can also represent relations.

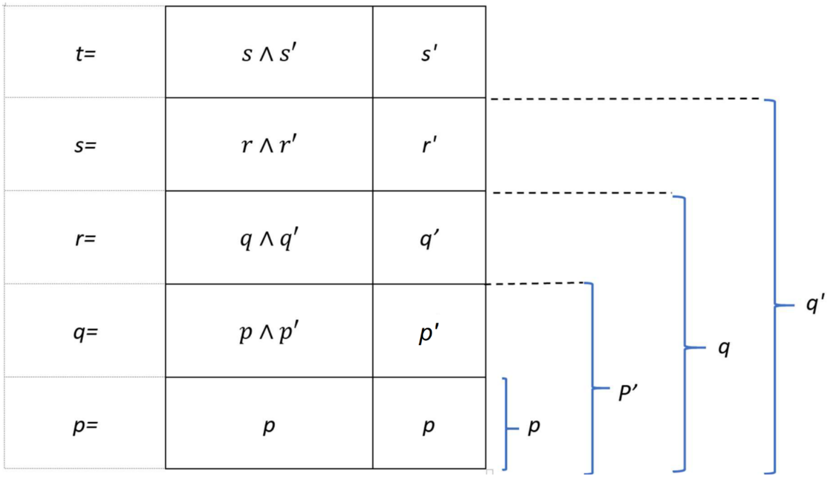
4.2.2. Form II.

Like the pair $\vee p$ and $\wedge \bar{p}$, $\wedge p$ and $\vee \bar{p}$ are also reversible operations of the elementary grouping, and Piaget (Piaget, Grize, 1972, pp. 327–329) based the second form of implications on this pair. As in Form I, relations between the propositions o, p, $\bar{p}$, and T are generalised to multiple propositions; however, Form II focuses on the conjunctive rather than the disjunctive compositions uniting propositions into a whole. With $\wedge p$ as the direct operation, a series of compound propositions can be constructed by composing propositions p, q, r, etc. with other propositions of the system p', q', r', etc. conjunctively to obtain the following equivalences: $p \wedge p' = q$; $q \wedge q' = r$; $r \wedge r' = s$; etc. (see Figure 2 on the next page).

Unlike Form I, Form II does not correspond to intrapropositional operations on classes. Grandchildren, for example, cannot simultaneously be siblings and their own first cousins since the intersection of complementary classes is empty. Nevertheless, elements of classes equivalent from one point of view may differ in degrees of a common property, thus allowing them to be ordered. Siblings A, B, C, etc., for example, differ according to age, and it is possible to order them via the order of birth without knowing their exact numerical ages: If A was born before B, A is older than B ($A \rightarrow B$), and if B was born before C, B is older than C ($B \rightarrow C$); clearly, A was born before C so that A is older than C ($A \rightarrow C$). In terms of propositions, let p represent "$A \rightarrow B$" and p' represent "$B \rightarrow C$", then q would represent "$A \rightarrow C$". In contrast to Form I, in which it is possible to infer q alone from either of the parts p or p' constituting it, neither p nor p' are sufficient by themselves to infer q in Form II. Just as it is not possible to infer A is older than C ($A \rightarrow C$) on the basis of either A being older than B ($A \rightarrow B$) or B being older than C ($B \rightarrow C$) alone, only p in conjunction with p' allows q to be inferred.

Figure 2

Grouping of Implications—Form II



Note. In the rows of the middle column, the compound propositions $p \wedge p$’, $q \wedge q$’, $r \wedge r$’, etc. are formed by conjunctions of the propositions in the rows immediately below it and the proposition to its right; for example, q (= $p \wedge p$’) is the conjunction of p, below, and p’, to the right; r (= $q \wedge q$’), of q (= $p \wedge p$’) below, and q’ to the right; etc. (Piaget, Grize, 1972, Fig. 47).

Conversely, maintaining “$A \rightarrow C$” (q) while denying either “$B \rightarrow C$” (p’) or “$A \rightarrow B$” (p) would be contradictory since it affirms the whole relation while denying one of its constituent parts. Analogously, $p = q \wedge \overline{p}$’ and p’ $= q \wedge \overline{p}$ would simultaneously assert the truth of q and the falsity of one of its constituent parts since $q = p \wedge p$’. The inverse operation used in Form I of the interpropositional grouping cannot, therefore, serve as an inverse operation in this form. The disjunctive composition of the negation of a proposition ($\vee \overline{p}$) with another proposition of the system, on the other hand, can, and compositions with this operation are $q \vee \overline{p}$’ $= p$; $q \vee \overline{p} = p$’; $\overline{p} \vee \overline{p}$’ $= \overline{q}$, etc., for example.

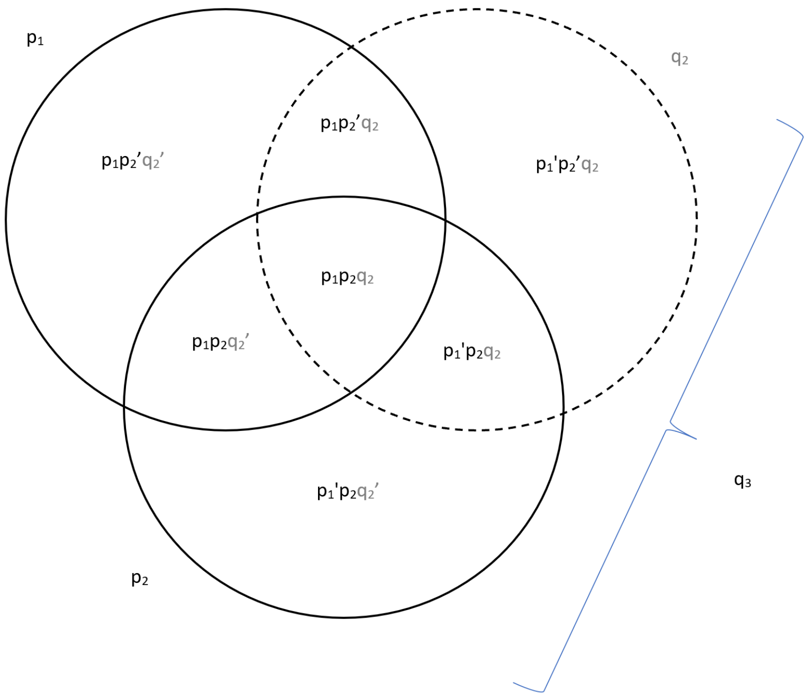
4.2.3. Form III.

In Form I, the wholes are constituted by exclusive disjunctions $p \vee p$’ $= q$, $q \vee q$’ $= r$, etc. The parts constituting the whole, therefore, have nothing in common. In Form II, on the other hand, the wholes are constituted by a conjunction $p \wedge p$’$= q$, $q \wedge q$’ $= r$, etc.; the whole is therefore constituted by what its parts have in common. Whilst Forms I and II are both extensions of the elementary grouping, they also represent extremes since the wholes are comprised of either entirely disjoint, Form I, or entirely conjoint, Form II, propositions. By constituting the wholes with propositions that are neither entirely disjoint nor entirely conjoint, Form III lies between these extremes.

Let propositions p_1 and p_2 constitute the whole q_1 via a non-exclusive disjunction $p_1 \vee p_2 = q_1$. Like the previous forms, Form III also introduces new implications: $\bar{p}_1 \supset p_2$ and $\bar{p}_2 \supset p_1$ (see Figure 3).

Figure 3

Grouping of Implications—Form III



Note. q_1 is the whole constituted by the non-exclusive disjunction of two propositions p_1 and p_2, and it is comprised of the three disjoint parts p_1p_2', p_1p_2 and p_1'p_2, where p_1' = $\bar{p}_1 \wedge q_1$ and p_2' = $\bar{p}_2 \wedge q_1$. Similarly, r_1 is not included in the diagram, but it designates the whole constituted by the non-exclusive disjunction $q_1 \vee q_2$, and is therefore comprised of the 7 disjoint parts indicated. The shade of the font indicates the origin of the contributions of the parts. Although the hierarchy of nesting propositions continues, a two-dimensional representation of their partitions has reached its limit (for a schematic representation, see Piaget, Grize, 1972, Fig. 48). $q_3 = p_2 \vee q_2$ does not belong to the hierarchy directly; however, it highlights a part-whole relation inherent in the nesting hierarchy of propositions that is relevant to the axiomatisation of propositional logic.

By defining p_1' as the proposition ($\bar{p}_1 \wedge q_1$) and p_2' as ($\bar{p}_2 \wedge q_1$), i.e., as relative complements, the grouping is as follows:

(i) The direct operation constitutes the following nesting wholes:

$(p_1 \vee p_2) = q_1$
$(q_1 \vee q_2) = (p_1 \vee p_2 \vee q_2) = r_1$
$(r_1 \vee r_2) = (p_1 \vee p_2 \vee q_2 \vee r_2) = s_1$
$(s_1 \vee s_2) = (p_1 \vee p_2 \vee q_2 \vee r_2 \vee s_2) = t_1$, etc.

Each of these wholes is composed of three disjoint parts (see Figure 3):

$q_1 = (p_1 \wedge p_2) \vee (p_1 \wedge p_2') \vee (p_1' \wedge p_2)$
$r_1 = (q_1 \wedge q_2) \vee (q_1 \wedge q_2') \vee (q_1' \wedge q_2)$, etc.

(ii) And conjunctions of negations of these parts constitute inverse operations, e.g.:

$p_1 = q_1 \wedge \overline{(p_1' \wedge p_2)}$; $p_2 = q_1 \wedge \overline{(p_1 \wedge p_2')}$

(iii) The general identity is, for example:

$q_1 \wedge \bar{q}_1 = o$; i.e., $(p_1 \vee p_2) \wedge \overline{(p_1 \vee p_2)} = (p_1 \vee p_2) \wedge (\bar{p}_1 \wedge \bar{p}_2) = o$

(iv) The special identities are, for example:
$p_1 \vee p_1 = p_1$; $q_1 \vee q_1 = q_1$; $p_1 \vee q_1 = q_1$

Piaget illustrates Form III in analogy with classes. Let P_1 be a class of blood relatives and P_2 be a class of relatives by marriage. Forming the union of $P_1 \cup P_2 = Q_1$, an individual belonging to Q_1 can be a blood relative and an in-law or one without the other. p_1 = "$x \in P_1$", p_2 = "$x \in P_2$" and q_1 = "$x \in Q_1$" express memberships propositionally, and, one of the members of Q_1 marrying, a new class of in-laws Q_2 is formed, in which some are blood relatives and in-laws while others are one without the other. The corresponding proposition is q_2 = "$x \in Q_2$", and $(q_1 \vee q_2) = r_1$ represents the union of these classes $Q_1 \cup Q_2$. Continuing in this vein, classes corresponding to s_1, t_1, etc. can be constructed.

Piaget highlights some implications in Form III and draws particular attention to one by defining q_3 as $p_2 \vee q_2$ (see Figure 3). In terms of the corresponding classes, it is clear that $P_2 \subset Q_3$. Although $P_2 \cup P_1$ is no longer included in Q_3, it is nevertheless included in $Q_3 \cup P_1$, the enlargement of Q_3 by the same class P_1; hence $(P_2 \cup P_1) \subset Q_3 \cup P_1$ provided $P_2 \subset Q_3$. Translated into propositions, $p_2 \vee p_1 \supset q_3 \vee p_1$ provided $p_2 \supset q_3$, and, through suitable substitutions, this formula is recognizable as $(p \supset q) \supset [(r \vee p) \supset (r \vee q)]$, axiom IV of Bertrand Russell's axiomatisation of propositional logic. According to Piaget, the special identity of the grouping $(p \vee p) = p$ also comes to expression in $(p \vee p) \supset p$, axiom I; axiom II, $p \supset (p \vee q)$, expresses the inclusion of parts in the whole

$(p \vee p') = q$ as well as special identities due to absorption $(p \vee q) = q$; and axiom III, $(p \vee q) \supset (q \vee p)$, expresses the commutativity of the operation $\vee$, on which the Forms I and III of the grouping rest. The Forms I and III of the grouping of implications thus condense the axioms of propositional logic, according to Piaget (Piaget, Grize, 1972, p. 331).

4.2.4. Form IV.

Although the forms of implication already presented are sufficient for an axiomatization of propositional logic, there is nevertheless a fourth form (Piaget, Grize, 1972, pp. 331–334). In Form I, the wholes $q = p \vee p'$, etc. are comprised of two disjoint parts; in Form III, on the other hand, the wholes are comprised of three disjoint parts $q = (p_1 \wedge p_2) \vee (p_1 \wedge p_2') \vee (p_1' \wedge p_2)$. Form IV complicates matters still further by adding yet another disjoint part $(p_1' \wedge p_2')$ so that the whole is now constituted by four disjoint parts:

$$q = (p_1 \wedge p_2) \vee (p_1 \wedge p_2') \vee (p_1' \wedge p_2) \vee (p_1' \wedge p_2').$$

However, Form IV is not just a complication for complication's sake. Class Q corresponding to the whole q in Form I has many alternative partitions; for example, Europeans (Q) are, from a German perspective, either Germans (P_1) or non-Germans (P_1'); from an Austrian point of view, on the other hand, they are Austrians (P_2) and non-Austrians (P_2'). Consequently, some Germans are non-Austrians and some Austrians are non-German. Since dual nationality is possible in the European Union, there are therefore German Austrians ($P_1 \cap P_2$), Germans who are not also Austrians ($P_1 \cap P_2'$), Austrians who are not also Germans ($P_1' \cap P_2$), and Europeans who are neither Austrian nor German ($P_1' \cap P_2'$). Analogously, four disjoint parts constitute the whole in Form IV: $q = (p_1 \wedge p_2) \vee (p_1 \wedge p_2') \vee (p_1' \wedge p_2) \vee (p_1' \wedge p_2')$. Furthermore, just as there are also Italians, Spaniards, Poles, Danes, Swedes, etc. in the EU each with their own national perspectives on Europeans, Form IV can also be extended to any number of propositions.

Form IV will be illustrated in the next section with two propositions, but the same rules of composition apply as the three preceding forms and the elementary grouping involving the affirmation and negation of a single proposition. Piaget thus concluded:

> There is nothing more, in fact, in these four forms than the progressive extension of the same operations $(\vee p)$ and $(\wedge p)$ hence one derives $(\wedge \bar{p})$ and $(\vee \bar{p})$: The Form II is the correlative[8] of Form I which itself extends directly [the elementary]

[8] According to Piaget (Piaget, Grize, 1972, pp. 256–257), the correlative and reciprocal operations can be derived from the inverse operation. $\bar{p} \wedge \bar{q}$ is the inverse of $p \vee q$, for example, and the operation involves two substitutions: conjunctions for disjunctions and vice versa, and affirmations for negations and vice versa. The outcome of performing just

grouping […]. Form III introduces two reciprocal implications there where Form I only knows one, and Form IV unites in a single whole all the operations developed in the preceding forms. There is thus only one grouping in four distinct forms, since the inverses, reciprocals and correlatives $(\vee p)$; $(\wedge \bar{p})$; $(\wedge p)$ and $(\vee \bar{p})$ are composable with each other. (Piaget, Grize, 1972, p. 333, my translation)

4.3. The Grouping of Binary Operators

Given a whole T that is partitioned dichotomously in two different ways by propositions p and q—$T = p \vee \bar{p}$ and $T = q \vee \bar{q}$, respectively—Form IV unites disjunctively the four parts engendered into a whole $T = (p \wedge q) \vee (p \wedge \bar{q}) \vee (\bar{p} \wedge q) \vee (\bar{p} \wedge \bar{q}) = (p * q)$. Although compound, the conjunctions are nevertheless propositions like any other; they can therefore be substituted for the propositions in Form I, and the substitutions constitute a grouping as follows (Piaget, Grize, 1972, Section 39 C):

Table 1

16 Distinct Logical Operators of Propositional Logic

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
pq	-	pq	-	-	pq	pq	-	pq	-	pq	-	pq	-	pq	-
$p\bar{q}$	-	$p\bar{q}$	-	$p\bar{q}$	-	-	$p\bar{q}$	$p\bar{q}$	-	-	$p\bar{q}$	$p\bar{q}$	-	-	$p\bar{q}$
$\bar{p}q$	-	$\bar{p}q$	-	$\bar{p}q$	-	$\bar{p}q$	-	-	$\bar{p}q$	-	$\bar{p}q$	-	$\bar{p}q$	$\bar{p}q$	-
$\bar{p}\bar{q}$	-	-	$\bar{p}\bar{q}$	$\bar{p}\bar{q}$	-	$\bar{p}\bar{q}$	-	$\bar{p}\bar{q}$	-	$\bar{p}\bar{q}$	-	-	$\bar{p}\bar{q}$	-	$\bar{p}\bar{q}$
$p * q$	o	$p \vee q$	$\bar{p} \wedge \bar{q}$	$p \mid q$	$p \wedge q$	$p \supset q$	$\overline{p \supset q}$	$q \supset p$	$\overline{q \supset p}$	$p \equiv q$	$p\ w\ q$	$p[q]$	$\overline{p[q]}$	$q[p]$	$\overline{q[p]}$

Note. The columns of this table are comprised of true conjunctions only (∧ is omitted to save space), and they are set out in pairs constituting the full complement of 4 conjunctions. Connecting the conjunctions in each column disjunctively generates the disjunctive normal forms of the logical operators in the bottom row. Except for *, *w*, *p*[*q*], and *q*[*p*] the binary operators are familiar. * denotes the complete affirmation; *w*, exclusive disjunction, and *p*[*q*] as well as *q*[*p*] are affirmations of *p* and *q*, in conjunction with either *q* and $\bar{q}$ or *p* and $\bar{p}$, respectively (based on Table 100 in Piaget, Grize, 1972, p. 214).

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the first substitution is the correlative $p \wedge q$; performing just the second operation, on the other hand, results in the reciprocal $\bar{p} \vee \bar{q}$. According to Halmos and Givant (1998, pp. 46–47), these operations are called "complement", "dual", and "contradual", respectively, and they depend on the principle of duality in a Boolean algebra. Moreover, these operations form a Klein four-group.

The logical operators of propositional logic can be expressed disjunctive normally as disjunctions of the conjunctions of Form IV. Via these disjunctive normal forms, Piaget shows that there are in fact 16 distinct binary operators (see Table 1). The columns of Table 1 are organised in complementary pairs with respect to the full complement of conjunctions, and, if the complementary pairs are composed disjunctively or conjunctively, the outcome is the complete affirmation or complete negation, respectively. These are the pendants to the laws of thought already highlighted in the elementary grouping (see Section 4.1), namely excluded middle and non-contradiction, respectively.

Table 1 sets out all 16 distinct logical operators, but Form IV is not simply a static taxonomy of logical operators. The interpropositional grouping is a system of transformations, and the logical operators can be transformed into each other as follows. Beginning with the equivalence $(p \wedge q) \vee (p \wedge \bar{q}) \vee (\bar{p} \wedge q) \vee (\bar{p} \wedge \bar{q}) = T$, e.g., the outcome of conjunctively composing the negation of the last conjunction with both sides of the equivalence is $[(p \wedge q) \vee (p \wedge \bar{q}) \vee (\bar{p} \wedge q) \vee (\bar{p} \wedge \bar{q})] \wedge \overline{(\bar{p} \wedge \bar{q})} = T \wedge \overline{(\bar{p} \wedge \bar{q})}$, i.e., $(p \wedge q) \vee (p \wedge \bar{q}) \vee (\bar{p} \wedge q) = \overline{(\bar{p} \wedge \bar{q})}$, which is equivalent to $(p \vee q)$ since $(p \vee q) = \overline{(\bar{p} \wedge \bar{q})}$. Algebraically, the transformation amounts to negating conjunctions of the complete affirmation and moving them to the opposite side of the equivalence, where they are composed conjunctively with the complete affirmation; for example, $(p \wedge q) \vee (\bar{p} \wedge q) \vee (\bar{p} \wedge \bar{q}) = T \wedge \overline{(p \wedge \bar{q})}$; i.e., $p \supset q = \overline{(p \wedge \bar{q})}$ (see Footnote 6b). By reversing the process, the original operator can then be restored. Moreover, $(p|q) \wedge \overline{(\bar{p} \wedge \bar{q})} = (p\, w\, q)$, for example, and $(p\, w\, q) \vee (\bar{p} \wedge \bar{q}) = (p|q)$. In other words, incompatibility $(p|q)$ is the outcome of composing reciprocal exclusion $(p\, w\, q)$ disjunctively with joint negation $(\bar{p} \wedge \bar{q})$. Whereas, the common part of an incompatibility $(p|q)$ and a disjunction $(p \vee q)$ $(= \overline{(\bar{p} \wedge \bar{q})})$ is $(p \wedge \bar{q}) \vee (\bar{p} \wedge q) = (p\, w\, q)$, a reciprocal exclusion. In short, the logical operators transform into each other, and the laws governing the system of transformations are those of a grouping. However, the conjunctions $\vee (p \wedge q)$ and $\wedge \overline{(p \wedge q)}$ rather than $\vee p$ and $\wedge \bar{p}$ of Form I constitute the direct and inverse operations of this manifestation of the grouping. And, the 16 logical operators defined disjunctive normally in Table 1 can be regarded as the operands of the grouping (cf. Seltman, Seltman, 1985).

The operations of the interpropositional grouping are as follows:

(i) The direct operation composes combinations of the four conjunctions constituting T disjunctively ($\vee$); e.g., $(o) \vee (p \wedge q)$; $(p \wedge q) \vee (p \wedge \bar{q})$.

(ii) The inverse operation is the negation of combinations of these conjunctions composed conjunctively ($\wedge$); e.g., $\wedge \overline{(p \wedge q)}$; $\wedge \overline{(p \wedge \bar{q})}$.

(iii) The general identity operation $\vee(o)$ leaves the elements it is composed with unaltered, e.g., $(p \wedge q) \vee (o) = (p \wedge q)$, and it is the product of the direct and inverse operations; e.g., $(p \wedge q) \wedge \overline{(p \wedge q)} = o$.

(iv) The special identities are:

a) Tautology: $(p \wedge q) \vee (p \wedge q) = (p \wedge q)$

b) Reabsorption: $(p \wedge q) \vee [(p \wedge q) \vee (p \wedge \bar{q})] = [(p \wedge q) \vee (p \wedge \bar{q})]$

c) Absorption : $(p \wedge q) \wedge (p * q) = (p \wedge q)$

(v) Associativity is again limited by the special identities.

In summary, this form of grouping engenders 16 distinct logical operators and unites them into a closed system of transformations. The interpropositional grouping thus represents operational transformations of a calculus of propositions, and, like the elementary grouping, the laws of thought are inherent in them; however, the transformations of the logical operators do not necessarily coincide with deductive inferences.

5. Implication, Transitivity, and Deduction

Via the direct operation of the elementary grouping, propositions p and $\bar{p}$ are composed disjunctively into a whole $p \vee \bar{p} = T$. The whole T is thus a proposition comprised of common $p \wedge T$ and non-common parts $\bar{p} \wedge T$ of p with T, i.e., $(p \wedge T) \vee (\bar{p} \wedge T) = T$. The fundamental operation of the elementary grouping thus engenders inclusions of parts in wholes. The conditional $p \supset T = (p \wedge T) \vee (\bar{p} \wedge T) \vee (\bar{p} \wedge \bar{T})$ is one of the few distinct logical operators already present in the elementary grouping. Since $\bar{T} = o$ therefore $\bar{p} \wedge \bar{T} = o$, $p \supset T$ converges with the affirmation $T[p] = (p \wedge T) \vee (\bar{p} \wedge T)$; the implications $p \supset T$, $\bar{p} \supset T$, $(p \vee \bar{p}) \supset T$, and $T \supset (p \vee \bar{p})$ are therefore expressions of the part-whole relations engendered by the fundamental operation of the interpropositional grouping. More generally, composing any two propositions x and y to form a whole z, $(x \vee y) = z$, via the direct operation of the interpropositional grouping generates relations of parts to the whole, which the following implications $x \supset z$; $y \supset z$; $(x \vee y) \supset z$ and $z \supset (x \vee y)$ express (Piaget, Grize, 1972, p. 343).

At this juncture, an ambiguity in Piaget's use of the term "implication" needs to be highlighted. In accordance with convention, Piaget uses "implication" and "conditional" synonymously to denote the logical operator. However, he also uses "implication" to denote the part-whole relations between propositions generated by the compositions of the interpropositional grouping. In such implications, the antecedent and consequent are related in some way. As a logical operator, implication $p \supset q$ is defined by $p \wedge q$, $\bar{p} \wedge q$, and $\bar{p} \wedge \bar{q}$ being true whilst $p \wedge \bar{q}$ is false. In part-whole relations on the other hand the truth of $p \wedge \bar{q}$ is excluded due to some relationship existing between the antecedent and consequent. For example, let p represent "$x \in$ mammals" and q, "$x \in$ vertebrates"; thus, some animals are mammalian vertebrates $p \wedge q$; some, non-mammalian vertebrates $\bar{p} \wedge q$; and others, neither mammalian nor vertebrate $\bar{p} \wedge \bar{q}$; however, invertebrates cannot be mammalian, so $p \wedge \bar{q}$ cannot be the case.[9] In this example, the antecedent and consequent are clearly related via their predicates, and Piaget (Piaget, Grize, 1972, pp. 226–227) distinguished implications referring to relations from implication as an operator and symbolised the former $p \longrightarrow q$.

According to Piaget (Piaget, Grize, 1972, p. 344), the primacy of implication is due to the transitivity of the nesting propositions it constitutes. But, first, referring to logical operators in general, surprisingly few are transitive. Piaget (Piaget, Grize, 1972, p. 340) illustrates intransitivity with mutual exclusions as follows:

$$(p|q) \wedge (q|r) \neq (p|r)$$
$$(p|q) = (\bar{p} \wedge \bar{q}) \vee (p \wedge \bar{q}) \vee (\bar{p} \wedge q)$$
$$(q|r) = (\bar{q} \wedge \bar{r}) \vee (q \wedge \bar{r}) \vee (\bar{q} \wedge r)$$
$$(p|q) \wedge (q|r) = (\bar{p} \wedge \bar{q} \wedge \bar{r}) \vee (p \wedge \bar{q} \wedge \bar{r}) \vee (\bar{p} \wedge q \wedge \bar{r}) \vee (\bar{p} \wedge \bar{q} \wedge r) \vee (p \wedge \bar{q} \wedge r)$$

Alternatively, it can be written in its dual form:

$$(p|q) \wedge (q|r) = (p * q * r) \wedge \overline{(p \wedge q \wedge r)} \wedge \overline{(p \wedge q \wedge \bar{r})} \wedge \overline{(\bar{p} \wedge q \wedge r)}$$

For example, let p = "$x \in$ invertebrate", q = "$x \in$ vertebrate", and r = "x lives attached to rocks (oysters, seaweed, etc.)". The five triple conjunctions $(\bar{p} \wedge \bar{q} \wedge \bar{r})$ = neither invertebrate, nor vertebrate, nor living attached to rocks; $(p \wedge \bar{q} \wedge \bar{r})$; etc. are all possible; in fact, only invertebrate vertebrates attached to $(p \wedge q \wedge r)$ or not attached to rocks $(p \wedge q \wedge \bar{r})$, and non-invertebrate vertebrates attached to rocks $(\bar{p} \wedge q \wedge r)$ are not possible. Moreover, it is clear that the incompatibility $p|r$ does not hold since there are some invertebrates that live attached to rocks $(p \wedge \bar{q} \wedge r)$. Several of the triple conjunctions are thus true due to $(p|q)$ and $(q|r)$, and, they are also true in $(p|r)$; however, $(p|r)$ does not necessarily follow from $(p|q)$ and $(q|r)$ since $(p \wedge \bar{q} \wedge r)$ is one of the triplets that is compatible with both $(p|q)$ and $(q|r)$ but not with $(p|r)$. For transitive logical operators, on the other hand, the conclusion would hold for all of the conjunctions compatible with the premisses. According to Piaget (Piaget, Grize, 1972, p. 345), *conclusive deductions* are based on the transitive logical operators.

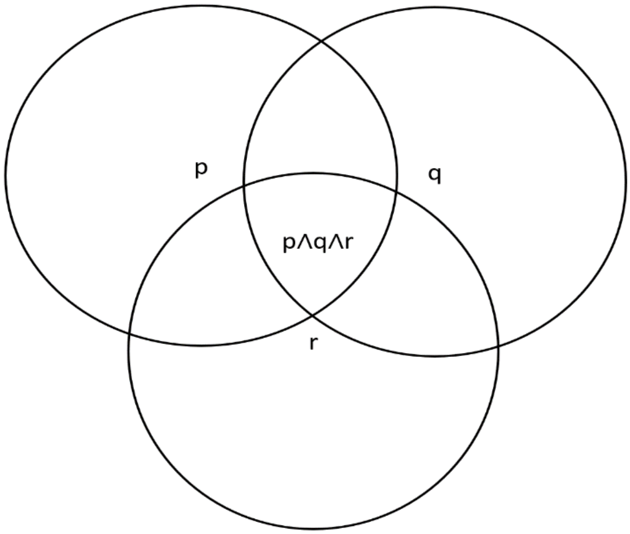
According to Piaget (Piaget, Grize, 1972, p. 344), conjunctions, implications, and equivalences, which are reciprocal implications, are the only transitive logical operators. In the case of transitivity of conjunctions $(p \wedge q) \wedge (q \wedge r) = (p \wedge r)$, let "$x \in$ both vertebrate and aquatic" $(p \wedge q)$ and "$x \in$ both aquatic and pulmonary" $(q \wedge r)$, for example, then, since whales, dolphins, etc. are pulmonary aquatic vertebrates, $(p \wedge r)$ is clearly true. However, $(p \wedge r)$ denotes all manner of pulmonary vertebrates, while $(p \wedge q \wedge r)$ only represents the small portion of them

[9] NB The conjunction $p \wedge q$ being true does not preclude the conjunctions $\bar{p} \wedge q$ and $\bar{p} \wedge \bar{q}$ also being true, although they are incompatible in classical logic (Apostel, 1982, Section 4).

inhabiting water. The transitivity of conjunctions is therefore founded on the three propositions *p*, *q*, and *r* having something in common (see Figure 4).

Figure 4

Transitivity of Conjunctions



Note. The intersections of three propositions *p*, *q*, and *r* are $(p \wedge q)$, $(q \wedge r)$, $(p \wedge r)$, and $(p \wedge q \wedge r)$ (based on Piaget, Grize, 1972, Fig. 51).

The operations of a grouping compose propositions, and Figure 4 represents a composition in which three propositions, *p*, *q*, and *r*, united into a whole $p \vee q \vee r$ are barely related. Nevertheless, they still have some common ground, and the transitivity of conjunctions bears on the minimum they still have in common: "the complete conjunction $(p \wedge q \wedge r)$ [which] is the lower bound (the greatest of the lower bounds) of *p*, *q* and *r*" (Piaget, Grize, 1972, p. 344). If, however, the propositions *p*, *q*, and *r* composing $p \vee q \vee r$ are related $p \supset q$ and $q \supset r$, then the clover-leaf shape of $p \vee q \vee r$ in Figure 4 takes on the form of a nesting hierarchy of inclusions like Figure 1. Transitivity is again due to what the propositions *p*, *q*, and *r* have in common; however, the common ground has now reached a maximum since the propositions are included in each other. This is the smallest of the unions three distinct propositions form, and, according to Piaget (Piaget, Grize, 1972, p. 344), it is an upper bound on transitivity. For equivalences $p \equiv q$, $q \equiv r$, then $p \equiv r$, on the other hand, the upper and lower bounds of the transitivity of propositions *p*, *q* and *r* coincide since $p \equiv q \equiv r$ (Piaget, Grize, 1972, p. 344). In short, transitivity is founded on what propositions have in common, and conjunctions, implications, and equivalences are the only transitive logical operators of the grouping, and implication marks the upper boundary of transitivity.

While transitivity is key to conclusive deductions, it is nevertheless quite rare among the logical operators of the interpropositional grouping. The operatory perspective on deduction might therefore appear to be inconsistent with actual deductive reasoning. However, the grouping unites logical operators into a closed system of transformations, and, via its operations, other operators interact with the few transitive operators. Given p, q, r, and $p \supset q$, for example, operators such as $(p \wedge r)$, $(q \wedge r)$; $(p \vee r)$, $(q \vee r)$; $(q|r)$; $(p|r)$; etc. are able to participate in the transitivity of implications; $(p \wedge r) \supset (q \wedge r)$; $(p \vee r) \supset (q \vee r)$; $(q|r) \supset (p|r)$; etc. therefore hold if $p \supset q$ holds. A richness of deductions commensurate with that of deductive reasoning is therefore generated by the many non-transitive operators participating in the transitivity of the few.

In summary, the operations of the interpropositional grouping compose propositions with one another, and some compositions engender part-whole relations between propositions. Implications as expressions of these part-whole relations thus go hand-in-hand with the fundamental operations of the interpropositional grouping. The conditional operator is one of the few operators already present in the elementary grouping, and the Forms I–IV of implication systematically extend the elementary grouping to multiple propositions by progressively differentiating the part-whole relations imminent in its propositions. These Forms thus propagate part-whole relations between propositions and thereby proliferate implications in the sense of relations. Moreover, transitivity is based on the nesting propositions engendered by the interpropositional grouping, and implication is not only one of the few transitive logical operators but also represents an upper bound on transitivity. Along with equivalence, which is in fact a double implication, implication is thus the primary source of conclusive deductions. In short, implication plays a fundamental role in the interpropositional grouping, and, along with equivalence, it accounts for the deductive fertility of this grouping (Piaget, Grize, 1972, p. 346).

6. The Nature of Deduction According to Piaget

The previous section has shown how the interpropositional grouping makes deduction possible. The nature of deduction is therefore tied up with the nature of the interpropositional grouping, and, in this section, I will attempt to shed light on the nature of deduction indirectly by characterizing the nature of the interpropositional grouping.

According to Piaget, the interpropositional grouping has synchronic and diachronic aspects. Starting with the latter, intelligence is a natural continuation of the biological adaptation of organisms (e.g., Piaget, 1952, Chapter Introduction; 1971b; 2001, Chapter 1). Organisms are open, self-regulating systems; as such, they are existentially dependent on their environments, and they strive to strike a balance between the demands of the environment on the one hand and the integrity of their biological organisations on the other through self-regulation. Like the biological organism, intelligence also has an internal organisation, adapts to its environment, and strives toward equilibrium; unlike the biological organism, though, intelligence actually achieves states of equilibrium.

Moreover, intelligence evolves in a sequence of stages over time (e.g., Piaget, 1977; 2001), and the sensorimotor, semiotic, concrete-operational, and formal-operational are the widely accepted stages, although their number varies in Piaget's works (Kesselring, 2009). These stages can be more broadly categorised into pre-operational—the sensorimotor, semiotic—and operational—concrete and formal—stages. As the terminology suggests, intra- and interpropositional operations occur at the operational stages.

The first cognitive equilibria are achieved at the operational stages, but they are presaged by coordinations of voluntary actions involving sensory stimuli and motor responses during the sensorimotor stage. The advent of language at the semiotic stage then heralds a change. The physical world constructed at the sensorimotor stage gradually becomes immersed in a world of representations. The effects of this immersion are twofold: on the one hand, the representational world not only captures the physical reality constructed at the sensorimotor stage but transcends it in all directions; on the other hand, the manipulations of objects, still enactive at the sensorimotor stage, can now be performed solely in the mind without physical manipulation accompanying them. The latter development is interiorization, and a whole new level of interiorization is achieved with the advent of operations (Piaget, 2001; Piaget, Grize, 1972, pp. 14–15).

Operations are interiorised actions, and just as actions occur in coordination with other actions, operations occur in concert with other operations. According to Piaget, equilibrium is achieved, however, when these operations are coordinated with others to form systems of transformations that are completely reversible. With the emergence of equilibria, the diachronic aspect is complemented by a synchronic aspect.

Turning to the synchronic aspect, operations in states of equilibrium form structured wholes amenable to formalisation, and psycho-logic models them using algebraic tools of logic. Groupings are thus formalisations of the structured wholes intra- and interpropositional operations form in states of equilibrium; as such they are new constructions, but they have functional roots in fundamental biological mechanisms (Piaget, Grize, 1972, pp. 14–15).

The biological roots come to expression in the cognitive function of the interpropositional grouping. Given two observable phenomena represented by propositions p and q, it is not immediately obvious how they are related to each other. Conjunctions $p \wedge q$, $\overline{p} \wedge q$, $p \wedge \overline{q}$, and $\overline{p} \wedge \overline{q}$ represent the four possible ways the phenomena can be associated in observation; however, individually each observation does not allow the relationship between the phenomena to be determined. Observation of p and q always occurring together, $p \wedge q$, for example, could mean that p and q are related in any of the 8 ways represented by the columns in Table 1 in which $p \wedge q$ occurs. Through observation of the occurrences of the four possible associations of p and q, on the other hand, the exact relationship between the phenomena can be determined. Observation of $p \wedge q$ and $\overline{p} \wedge \overline{q}$ occurring without exception but no cases of either $\overline{p} \wedge q$ or $p \wedge \overline{q}$, for example, indicates that the phenomena represented by p and q are equivalent; whereas observation of $p \wedge q$, $\overline{p} \wedge q$, and $\overline{p} \wedge \overline{q}$ but no cases of $p \wedge \overline{q}$ means that p implies q (see Table 1). The interpropositional grouping thus serves as a cognitive tool for determining connections between observable phenomena and therefore represents a cognitive adaptation to the environment.

According to Piaget, three key ideas characterise structures: "the idea of wholeness, the idea of transformation, and the idea of self-regulation" (Piaget, 1970, p. 5). Piaget draws attention to the relational nature of parts and whole in structures; however, the whole is neither the sum of its parts nor are the parts wholly determined by the whole. Neither the whole nor the parts are primary, and, instead of bottom-up or top-down constructions, the parts and whole are the outcome of laws of construction that are both structured and structuring. Moreover, the parts are transformed by the system's laws of composition, but the system of transformations as a whole is closed since the outcomes of these transformations also belong to the system and preserve its laws. In the interpropositional grouping of logical operators, for example, neither the operands, the 16 logical operators, nor the whole structure, the grouping, are primary; they are the outcome of interpropositional operations of thought achieving a state of equilibrium. Moreover, the operations of the grouping are laws of composition that transform the logical operators operated on, and the outcome of these operations is another logical operator that also preserves the laws of the system; the system of operations formed by the grouping is, therefore, closed and self-regulating. Like its namesake the group, the grouping of interpropositional operations thus fulfils Piaget's characterisation of a structure.

Since Frege, it has been standard practice to axiomatize logic in analogy with the substantial axiomatisations of extra-logical sciences (Hintikka, Sandu, 2007, Section 5). Accordingly, logic is reduced to a handful of axioms and rules of inference, from which all the formulae of logic can be derived. Axiomatisations of logic like those of extra-logical sciences are therefore systematisations of a theory; nonetheless, there are significant differences between the two. Substantial theories are sets of propositions that correspond to an extra-logical reality, and, by reducing these propositions to a handful of postulates from which those describing or predicting the targeted realities can be derived, axiomatisations assist in understanding these realities. Moreover, in the axiomatisation of substantial theories, the derivations correspond to what is ordinarily understood by deduction, and the correspondence of the theory with reality as well as verification of its predictions tend to transmit truth backwards to the axioms. Like substantial axiomatisations, the axioms of a formal system are the underived formulae on which the derivation of other formulae of the theory are founded. However, there is no difference in principle between the derived formulae and the axioms of a formal system—the latter are simply formulae without premisses.

Moreover, any set of formulae can serve as axioms as long as they are consistent—a formula and its contradictory cannot be derived from them—preferably independent, and semantically complete—all the true formulae of the theory can be derived from them. Moreover, in contrast to a substantial axiomatisation, the derivation of the formulae in formal systems need not correspond to the rules of inference in logic; a mechanical means of systematically listing all of the formulae is sufficient. Despite similarities with formal systems, the interpropositional grouping models an extra-logical reality; it is therefore a substantial axiomatisation, specifically a substantial axiomatisation of the equilibrium achieved by interpropositional operations of thought.

The interpropositional grouping models the reversibility of rational thought, on the one hand, and the systems of transformations operations of thought engender when they achieve equilibrium, on the other. According to Piaget (Piaget, Grize, 1972, Sections 36–38), the interpropositional grouping represents either a relaxation of the strict reversibility of groups through augmentation with inclusions and self-inclusions or a tightening of the operations of a lattice through the introduction of reversibility into its operations. Although not purely abstract, the interpropositional grouping is thus a mathematical structure that lies mid-way between groups and lattices. According to Grize (2013, p. 152), Piaget based the interpropositional grouping on Boolean structures.

There are many expressions for each of the different logical operators; despite their disparate guises, though, they can be shown to be equivalent by reducing them to their normal forms. In fact, Table 1 represents a classification of equivalent formulae via their disjunctive normal forms. $\{pq \vee \bar{p}q \vee p\bar{q}\}$ thus represents the class of formulae $p \vee q$, $p \vee q \vee q$, etc; $\{T\}$, the class of tautologies $p \vee \bar{p}$, $q \vee \bar{q}$, etc., and the class of contradictions $p \wedge \bar{p}$, $q \wedge \bar{q}$, etc., being empty, is represented by the null class $\{o\}$. Moreover, the propositional connectives $\vee$, $\wedge$, and $\bar{\ }$ are congruent with operations $\cup$, $\cap$, and ', respectively, on these classes (Rutherford, 1966, pp. 50–51). The operations of this grouping, therefore, correspond to operations on classes of the classification of formulae; transforming $p\,w\,q$ into $p \vee q$ via the direct operation $\vee pq$, for example, is $(p\,w\,q) \vee pq = (\bar{p}q \vee p\bar{q}) \vee pq = p \vee q$, which corresponds to $\{\bar{p}q \vee p\bar{q}\} \cup \{pq\} = \{\bar{p}q \vee p\bar{q} \vee pq\}$ in terms of classes; transforming $p \vee q$ back to $p\,w\,q$ via the inverse operation $\wedge \overline{(pq)}$, on the other hand, corresponds to the relative complement of $\{pq\}$ in $\{p \vee q\} = \{\bar{p}q \vee p\bar{q} \vee pq\}$, i.e., $\{\bar{p}q \vee p\bar{q}\} = \{p\,w\,q\}$. The general identity is composed of the direct and inverse operations of a grouping, and it leaves any element of the grouping unaltered $r \vee o = r$ but $r \wedge o = o$; in terms of the classification of formulae, $\{r\} \cup \{o\} = \{r\}$ and $\{r\} \cap \{o\} = \{o\}$; hence $\{o\} \leqq \{r\}$ for all $\{r\}$. T, on the other hand, is a special identity $T \vee r = T$ and $T \wedge r = r$; therefore $\{T\} \cup \{r\} = \{T\}$ and $\{T\} \cap \{r\} = \{r\}$; hence $\{r\} \leqq \{T\}$ for all $\{r\}$. In other words, all the classes in the classification of formulae include $\{o\}$ and are included in $\{T\}$; the classification therefore has a null $\{o\}$ and a universal element $\{T\}$. Moreover, each element r of the grouping has an inverse such that $r \wedge \bar{r} = o$ and $r \vee \bar{r} = T$; for every $\{r\}$ there is therefore a $\{s\}$ such that $\{r\} \cap \{s\} = \{o\}$ and $\{r\} \cup \{s\} = \{T\}$, i.e., a complement.

Since the operations of the grouping are also distributive, this complement is unique and can be denoted r'. The equivalence classes of formulae under the lattice operations corresponding to the operations of the grouping, therefore, constitute a complemented distributive lattice. Since a Boolean algebra is a complemented distributive lattice (Halmos, Givant, 1998; Rutherford, 1966), the structure Piaget loosely characterised as being mid-way between groups and lattices seems to correspond to a Boolean algebra.

Halmos (2016, Chapter Introduction Section 2) remarked that "Boolean algebras have an almost embarrassingly rich structure […]. In every Boolean algebra there is, moreover, a natural order relation [and] [t]he algebraic structure and the order structure are as compatible as they can be". It is therefore desirable to narrow down the nature of deduction still further.

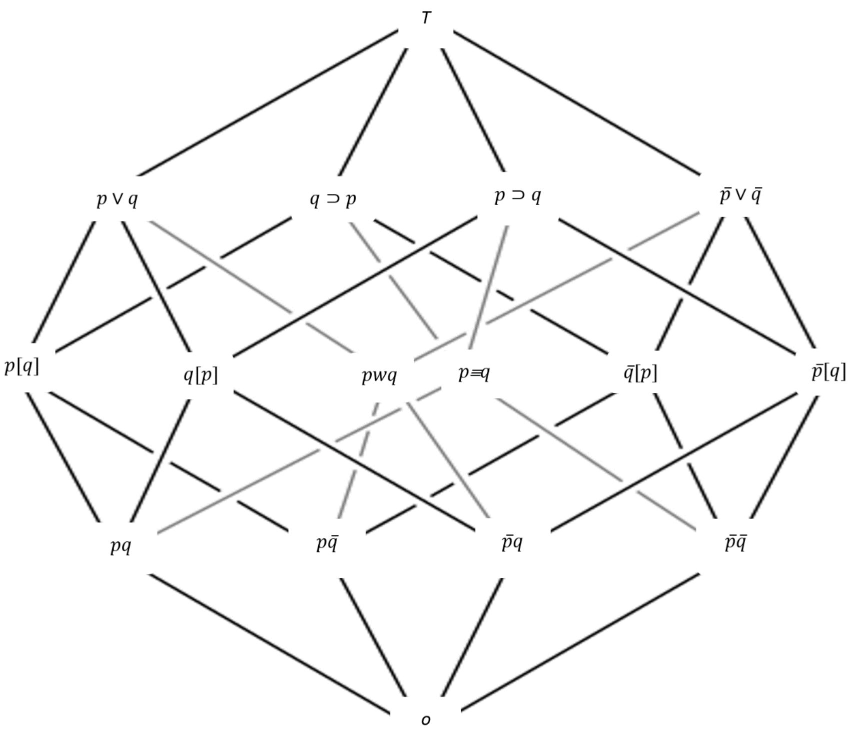
From the viewpoint of classical logic, $p \supset q$ has the same truth conditions as $\bar{p} \vee q$, namely, true except when p is true and q is false. Since arbitrary propositions may be substituted into the propositional variables p and q, it is not possible to preclude the falsity of the compound proposition without imposing some additional constraints. In a free Boolean algebra, the postulates constitute the only constraints on propositions. To determine the additional constraints on p and q necessary for $p \supset q$ to be true without exception, consider any two propositions p and q belonging to the classification of propositional formulae in Table 1 and the class $\{p\}' \cup \{q\}$, which corresponds to composing q disjunctively with the negation of p, $\bar{p} \vee q$. Clearly $p \supset q$ is also a member of the class $\{p' \cup q\}$, but, for it to be true without exception, $\{p\}' \cup \{q\} = \{T\}$, i.e., $\{q\} \geqq \{p\}$ so that $\{q\}$ is in the interval $[\{p\}, \{T\}]$ (Rutherford, 1966, pp. 51–52).

Figure 5 (on the next page) is a Hasse diagram of the equivalence classes in Table 1. Although it is simply an alternative representation, it has the advantage of bringing the lattice structure clearly to the fore. Referring to Figure 5, the condition set out in the previous paragraph is fulfilled provided $\{q\}$ is a class of propositions occupying a node on one of the lines connecting $\{p\}$ with $\{T\}$; for example, $p[q]$, $q[p]$, $p \vee q$, etc., are propositions belonging to classes on the line connecting the class $\{p \wedge q\}$ with $\{T\}$; the implications $p \wedge q \supset p[q]$, $p \wedge q \supset q[p]$, $p \wedge q \supset p \vee q$, etc. are therefore tautologies. From the viewpoint of Table 1, $p \wedge q$ implies all those logical operators in which it is affirmed in the disjunctive normal form.

In Section 5, I mentioned how Piaget distinguished between implications as relations and as operators. In essence, relations in contrast to operators cannot be false due to some relation existing between the propositions, and I illustrated the difference with an implication in which the antecedent and consequent are related via their predicates. By means of the lattice structure, it is possible to deal with such relations more generally. On the one hand, if $\{q\}$ is in the interval $[\{p\}, \{T\}]$, $p \supset q$ is a tautology, and, in Piaget's terminology, it is an implication in the relational sense. Moreover, the order relation between the classes is $\{q\} \geqq \{p\}$, which is also known as an inclusion relation since it is equivalent to

Figure 5

Hasse Diagram of the 16 Logical Operators of Propositional Logic



Note. The figure represents the projection onto the plane of a four-dimensional cube. The logical operators occupy the points of intersecting lines, and lines connecting points represent inclusion relations. Thus $p \supset q \geqq p \equiv q$, $q[p]$, $\bar{p}[q]$, pq, $\bar{p}q$, $\overline{pq}$ and o; but not $p\bar{q}$ (after Rutherford, 1966, Fig. 7).

$\{p\} = \{p\} \cap \{q\} \equiv \{p\} \cup \{q\} = \{q\}$. The same inclusion relation can also be expressed, admittedly less conventionally, in terms of parts and wholes. In Piaget's parlance, then, $\{p\}$ being a part of the whole $\{q\}$, i.e., $\{p\} = \{p\} \cap \{q\}$ or equivalently $\{p\} \cup \{q\} = \{q\}$, thus refers to an implication $p \rightarrow q$ that cannot be false because an inclusion relation $\{q\} \geqq \{p\}$ exists between the antecedent and consequent. Moreover, by generalising the elementary interpropositional grouping formed by the affirmation and negation of a single proposition p to multiple propositions, Piaget, in effect, inserted propositions q, r, s, etc. in the interval $[\{p\}, \{T\}]$ of the elementary grouping. The implications $p \supset q$, $q \supset r$, etc. engendering the forms of implication are thus implications in the sense of relations $p \rightarrow q$, $q \rightarrow r$, etc., and Piaget's allusions to part-whole relations in describing these implications seem in fact to correspond to inclusion relations (Winstanley, 2021, Section 3.1).

Piaget discerned four different forms of implication, but only Forms I–III give rise to conclusive deductions. Developmentally, the interpropositional grouping synthesises intrapropositional groupings of operations on relations and classes into a single structure, and Forms I and III of implication can be modelled by operations on classes, whereas operations on relations model Form II. Moreover, the part-whole relations between propositions are the basis for deduction in Forms I and III; deduction in Form II on the other hand is based on the transitivity of the order relation. Lattices have two equivalent definitions (see Footnote 3), one emphasising operations; the other, being based on a poset, highlighting their relational nature. Moreover, they are connected by the identity $y = x \cap y \equiv x \geqq y \equiv x \cup y = x$. Order and inclusion relations, two seminal characteristics of lattices, are therefore inherent in the Forms I–III of implication. Piaget thus appears to have attributed the nature of deduction specifically to the lattice structure inherent in the embarrassing richness of a Boolean algebra.

Moreover, Piaget attributed the deductive richness of reasoning to propositions participating in the transitivity of logical operators like implication via the operations of the grouping. With the help of lattice theory, this can be circumscribed precisely: “The totality of valid deductions from a proposition or set of axioms p are […] those propositions belonging to the classes of the interval $[[\{p\}, \{T\}]]$” (Rutherford, 1966, p. 52). According to Piaget, From IV does not give rise to any new implications; however, as part of the algebraic rather than the order structure of a Boolean algebra, it can nevertheless contribute to the deductive richness of reasoning.

7. Conclusion

According to Piaget, the nature of human propositional reasoning lies in the interpropositional grouping, the calculus embodied in propositional operations, and the nature of propositional deduction, in particular, lies in the relations between propositions inherent in the Forms I–III of implication. If the interpropositional grouping constitutes a Boolean algebra, as I have argued, then the nature of deduction lies specifically in the order rather than the algebraic structures of this embarrassingly rich structure. I therefore conclude that the nature of deduction according to Piaget lies specifically in the lattice engendered by the operations of the interpropositional grouping.

Finally, having characterised the nature of deduction, it would be remiss not to touch at least briefly on its implications for logic. How Piaget regarded the relationship between the forms of implication and axiomatisations of propositional logic was touched on briefly at the end of Section 4.2.3. Put succinctly, the interpropositional grouping is the natural structure inherent in propositional reasoning, which “lies ‘beneath’ the operations codified by axioms [of logic]” and furnishes “the underpinnings of logic” (Piaget, 1970, p. 31). In other words, the interpropositional grouping forms the foundation for propositional logic. However, propositional logic is not synonymous with the interpropositional grouping. According to Piaget, "logic is the mirror of thought, and not vice versa" (Piaget, 2001, p. 27), and, after several iterations, Piaget eventually defined logic without the aid of metaphor as "the formal theory of deductive operations" (Piaget, Grize, 1972, p. 20, authors' emphasis). Piaget's psychological theory of propositional reasoning therefore forms an evidential basis for a logic conceived as a formal theory (Winstanley, 2021), and the forms of implication will clearly play a seminal role in its construction. To my knowledge, a Piagetian logic has yet to be constructed (Apostel, 1982; Grize, 2013); if it were, however, it would arguably constitute a natural logic among the plurality of logics since a logic is imminent in a structure (Shapiro, 2014), and the logic imminent in a natural structure like the interpropositional grouping would constitute a natural logic.

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