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Article

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AGAINST VACUISM

SUMMARY: This paper discusses the question of whether all counterfactuals with necessarily false antecedents (counterpossibles) are vacuously true. The orthodox view of counterpossibles (vacuism) answers that question in the affirmative. This paper explains vacuism before turning to examples from science that seem to require us to reason non-trivially using counterpossibles, and it seems that the counterpossibles used in such cases can be true or false. This is a threat to vacuism. It is then argued that the same kind of reasoning which produces non-trivial counterpossibles in scientific cases can be extended to the case of counterpossibles in mathematics. Ordinary counterfactual reasoning relies on rejecting background assumptions in order to assume the truth of the antecedent. A failure to perform this process in the counterpossible case is what leads one to vacuism and it is explained how this process produces non-vacuous; counterfactuals, scientific counterpossibles, and mathematical counterpossibles.

KEYWORDS: counterfactual, counterpossible, vacuism, non-vacuism, impossible worlds.

1. Introduction

Orthodoxy states that a counterfactual ($A > B$) is true when the nearest A-worlds are also B-worlds. For any counterfactual with an antecedent that logically implies a consequent, the counterfactual will come out true, regardless of the content of either part. If there are no A-worlds as described by the antecedent, then trivially *all* A-worlds are B-worlds, i.e., the counterfactual will come out as true (Stalnaker, 1968). Counterpossibles are a subset of counterfactuals that con-

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tain an impossible antecedent. For the purposes of this paper, we can assume that impossibility to be of the highest level. I will simply assume for now that this is the level of metaphysical necessity. So the counterpossibles I will generally be concerned with will be those with a metaphysically impossible antecedent, I will symbolise these as $A_i > B$. In virtue of being metaphysically impossible, it seems that there are no worlds at which A_i will be the case, so the orthodoxy tells us that any such counterpossible will come out as trivially true, regardless of subject matter. This theory is known as vacuism, one key proponent of vacuism is Williamson (2007; 2018). This paper mainly addresses his formulation of vacuism and his arguments for it, ultimately arguing that some counterpossibles are non-trivial.

Of course, no non-vacuists need to say that all counterpossibles are non-trivial, so many restrict the non-triviality thesis to specific domains. One place that it might be difficult to imagine the occurrence of non-trivial counterpossibles is in mathematical reasoning. Proofs by reductio seem to typically involve making impossible suppositions and then reasoning from them, ultimately proving that indeed the supposition is impossible and necessarily false. For these to work, it seems that all the statements in these proofs need to be true. This is exactly as the vacuist prescribes and so one might view this as a compelling argument to agree with vacuism. I disagree, and I think that the reasons we can give for believing in the non-triviality of other counterpossibles are extendable to the case of non-trivial countermathematicals. The basic argument I will offer is as follows: We have compelling reasons to think that there are non-trivial counterpossibles in the sciences, some scientific counterpossibles come out as false (and some true). This datum is significant enough to override the prescriptions of logical orthodoxy. Two things might be going on at this stage, either: we are implicitly using a non-standard semantics for counterfactuals in these cases, allowing them to come out with differing and non-vacuous truth values or; we are working within a standard semantics but still delivering this verdict, contra orthodoxy. It seems most likely that a vacuist would say such counterpossibles are true because there are no A_i worlds. It further seems that what might actually be going on in the cases of scientific counterpossibles is that we are genuinely considering an impossible world, and because the truth value of B is up for grabs at these A_i worlds, the truth value of the counterpossible as a whole can change. I will discuss how this is applicable to the case of countermathematicals.

This is the strategy I will be considering in this paper. We should genuinely consider the closest world at which any A_i is the case. Considering impossible worlds, on some minimal level allows us to deliver the verdict from science, it also shows us that vacuism is false. The unique contributions this paper aims to make lie in several places. As above, this paper aims to show that if we genuinely consider an impossible world/suppose that A_i, then different counterpossibles will have different truth values. This is illustrated in the cases of scientific counterpossibles discussed. This paper also aims to show that vacuism about counterpossibles in mathematics is a redundant thesis. Further contributions to the literature are made by distinguishing between two kinds of projects that one might undertake in counterfactual form. In the first case, one may wish to use counterfactual form to work out the truth value of the statement which forms the antecedent. The second case involves reasoning from the antecedent to potential consequents to see what would be the case, if the antecedent were true. Importantly for this second process, this is done regardless of the actual truth value of the antecedent, one has to genuinely consider it/suppose it to be true (Section 3.4 of the current paper). This distinction is a close companion of the distinction between a consensus and non-consensus context given by Yli-Vakkuri and Hawthorne (2020).[1] This paper aims to show that Williamson is engaged in the first kind of process, rather than the second kind of process. Even if all counterpossible statements in the first kind of process turn out to be true, it is not the case that counterpossibles used in the second process will, so vacuism is false. What Williamson (2007; 2018) does is to determine the truth value of a statement (A_i), which he does by embedding it as the antecedent in a counterfactual form. But this is different from genuinely considering what would be the case if A_i were true. Importantly, this genuine consideration is what Brogaard and Salerno (2016) are engaged in when responding to Williamson and this is the core reason that Brogaard and Salerno appear to be in disagreement with Williamson. They each think the other side is performing the same reasoning task and producing a different result, when in fact they are engaged in different enterprises. So this paper provides a methodological explanation of why the disagreement between vacuists and non-vacuists has arisen. It is also worth noting that, in the literature on counterpossibles, it is often the case that non-vacuists will provide examples of counterpossibles that are non-vacuous (e.g., Jenny, 2018), but not necessarily provide a general overarching explanation for their non-vacuity. They say that the counterpossibles in question are non-vacuous, but not always why. This paper aims to start providing an answer to that question by pointing to the use of non-vacuous counterpossibles in scientific explanations, and showing how the mathematical cases mirror this.

As a final prelude before starting the discussion, it will be worth clarifying some assumptions at play. It is worth stating up front that I am implicitly assuming some variation of a Lewisian conception of worlds,[2] which includes an account of impossible worlds. Although I have some reservations about the specific account, Yagisawa's (2010) extended modal realism is an interesting take on impossible worlds and the general spirit of that account can be kept in mind when impossible worlds are mentioned in this paper. Given an account of both possible and impossible worlds, I think it is very plausible that one can maintain the standard semantics of Lewis-Stalnaker, because if there are impossible worlds, then we can assess counterpossibles on the basis of the closest one.

[1] This also seems close to the suppositional procedure that Williamson describes in Suppose and Tell (2020). However, as will be argued for later on, I think Williamson fails to properly engage in the suppositional procedure, and that is why he believes the counterpossibles to all be true.

[2] Along with the associated semantics.

However, it is also worth noting that is obviously not an inherent commitment of non-vacuism, one can be a non-vacuist without believing in this specific conception of impossible worlds. One need not even accept impossible worlds at all, perhaps one way to do this is to alter the standard Lewis-Stalnaker semantics instead.[3] Although one might say that an appeal to either a different ontology of possible worlds or a different semantics is problematic, it is worth noting that the only reason that Williamson thinks he can achieve a vacuist result is by assuming a specific semantic account/a specific conception of worlds, so if this is a problem for non-vacuists, it is equally a problem for vacuists. One way to read the following arguments about scientific and mathematical practice and the treatment of counterpossibles is that they provide reasons to think that experts in those disciplines make assumptions close to the ones described above, and that provides us a reason to make them too, rather than the ones that vacuists make. With these clarifications in place, we are in a position to begin considering counterpossibles.

2. Counterpossibles in Science

2.1. Tan's Cases

There are a plethora of examples of counterpossible pairs that intuition tells us have different truth values. But intuition only takes us so far, the vacuist can simply say this is the appearance of the distinct truth values, but the logical form tells us we are actually mistaken. This response by the vacuist will not work in the scientific case. If good scientific practice leads us to assign some counterpossibles as being false, we need to account for this. The results from science outweigh philosophical/logical inclinations we may have. Compare this with how developments in quantum mechanics have led some to alternative quantum logics to account for the discrepancies (e.g., Putnam, 1969), of course such usages are controversial and by no means the orthodox, but this shows that it is not universally agreed that classical logic always has the correct verdict. The usage of counterpossible reasoning in the sciences is documented by a number of people (McLoone, 2020; Wilson, 2021). One such discussion takes place in Tan (2019), in which he presents examples of the use of non-trivial counterpossibles in science. Not only are there multiple examples of counterpossibles used in science, but they are used in different ways and for different purposes. Tan (2019) focusses on their use in: scientific explanation; idealised scientific models; and in reasoning about superseded scientific theories. In each of these cases, he offers an archetypal example of a counterpossible and discusses why viewing it as counterpossible and as non-trivial is the correct verdict. In the case of scientific explanation, the counterpossible offered is:

[3] Another way to do this would be to adopt some appropriate form of non-classical logic. Whilst I do not wish to rule out this route, it will not be discussed in this paper.

(A) “If diamond had not been covalently bonded, then it would have been a better electrical conductor” (Tan, 2019, p. 40).

Tan claims that this is a scientific explanation of the fact that diamond cannot conduct electricity whereas solid carbon in some other forms can. The reason the covalent bonding explains this fact is because covalent bonds do not leave free electrons, as they “use up” all the electrons forming the strong bond. In other substances, free electrons allow for electrical conductivity (Tan, 2019, p. 40). The property of poor conductivity that diamond has is brought about as a result of these bonds, and so the microphysical structure. This counterfactual then provides an explanation in virtue of highlighting that dependence relation. But one might wonder if this is indeed a counterpossible; one may wonder whether diamond could have been otherwise bonded, in which case this would be a mere straightforward counterfactual. One can approach this in two ways, we might consider whether something is called diamond in virtue of its microphysical structure or in virtue of its theoretical role in science (Tan, 2019). Going the first route, one can easily see that this is a counterpossible, because if something is only diamond in virtue of its microphysical structure, then something which had a different microphysical structure would not be diamond. As a matter of metaphysical necessity, diamond has the structure that it does. So it is metaphysically impossible for diamond to be differently bonded.

Going the second way, one may think that we define diamond by its theoretical role, the diamond-stuff is the stuff that does *x*, *y* and *z*. But the reason diamond is distinguished from other substances, and the reason it does the things it does, is because of its microphysical structure. In other words, nothing else could do the things diamond does without its microphysical structure. Nothing could fill the diamond role without actually being diamond. So again, it is metaphysically impossible that diamond could have been differently bonded than it in fact is. So it seems then, that statement A above is a counterpossible. Tan (2019) goes further than this, he insists that this is also a counterpossible which is true, and non-vacuously so. This is because it describes an empirical fact, that the poor conductivity of diamond physically depends on its microphysical structure. So science relies on non-vacuous counterpossibles in scientific explanation (2019, p. 42). One can easily see how this is not an isolated case because many scientific explanations of why substances have the properties they do will rely on a similar explanatory structure.

As stated, Tan also thinks that we need to make use of substantially true counterpossibles when reasoning about superseded scientific theories. Sometimes, we need to reason about scientific theories using counterfactuals; “If Jupiter were a point mass then…” and “If classical mechanics had been true…” are examples of each of these (Tan, 2019, p. 48). As Tan points out, we might counterfactually reason about a false theory to describe its empirical content, e.g., “had the geocentric Ptolemaic system been correct, celestial spheres would be unobservable entities”. Counterfactual reasoning is also used in order to explain the falseness of a false scientific theory. Tan considers a straightforward example of this concerning Bohr's theory of the atom (Tan, 2019, p. 48):

(B_1) If Bohr's theory of the atom had been true, then an electron's angular momentum, *L*, in the ground state would have been observed at $L = h$ (the reduced Planck constant).

(B_2) It is not the case that the electron's angular momentum, *L*, in the ground state is observed at $L = h$.

(B_3) Therefore, Bohr's theory of the atom is false.

Bohr's theory of the atom predicts/requires that the angular momentum of an electron is observed in the above way, that is to say that (B_1) is correct. Given that that is a result of the theory, if the theory were correct then that would be the case. Repeated experimentation and observation has shown that the angular momentum of an electron is actually zero in the ground state, i.e., (B_2) is true. Given that both (B_1) and (B_2) are true, it then simply follows that (B_3) is true. This is a substantial result, and clearly (B_1) is true more than merely trivially. As Tan puts it: "in order for this commonplace pattern of reasoning to be epistemically fruitful, theory-evaluating conditionals must describe genuine relations of counterfactual dependence and implication. They must, in other words, be non-vacuously true" (Tan, 2019, p. 49).

This seems to be correct, the above essentially takes the form of "if *that* were right, we would see *this*. We do not see *this*, so *that* must be wrong". We want such arguments to produce truth that is not merely trivial, because the process Tan talks about seems like an example of good scientific reasoning. There are many examples of this process being used in the sciences for all manner of theories. As a method of theory falsification, it is a good one, and we need it to produce substantive, non-trivial results. Now one may be willing to accept this but unwilling to extend it to the counterpossible case, because of a commitment to vacuous counterpossibles. The problem here is that (B_1) is already a counterpossible. This archetype of non-vacuous scientific reasoning turns out to involve counterpossible reasoning. If one wishes to trivialise all counterpossibles then one is going to have to trivialise a lot of scientific reasoning, and this seems an unattractive feature of any account. The reason that (B_1) is a counterpossible is that Bohr's theory of the atom is an inconsistent theory. It rests on both classical and quantum assumptions, therefore some aspects of the theory represent orbiting electrons as radiating energy as they move about; other aspects of the theory represent electrons as non-radiative (Tan, 2019, p. 49). In other words, the theory as a whole contains a contradiction, as it represents electrons both as radiating energy and as not radiating energy. (B_1) does not merely refer to one aspect of Bohr's theory, it refers to the theory as a whole, and the theory as a whole contains this contradiction. So it is simply logically impossible that Bohr's theory of the atom be true, it is impossible that Bohr atoms could exist. (B_1) then, is a counterpossible. But we have already established that (B_1) is non-vacuously true. A potential response from vacuists could be that we can maintain vacuism because we accept that (B_1) is true (vacuously) and also accept that (B_1*) is true:

(B_1*) If Bohr's theory of the atom had been true, then an electron's angular momentum, L, in the ground state would *not* have been observed at $L = h$ (the reduced Planck constant).

(B_1*) negates the consequent of (B_1), but as it is a counterpossible, is also true (vacuously so). The vacuist might respond that the reason we appeal to (B_1) rather than (B_1*) is because the former has proved useful for scientific progress and prediction due to the way the world happens to be, whilst the latter has not. The problem I see with this response is that I do not think particle physicists would accept that (B_1) and (B_1*) are equally true. It seems much more likely that physicists would judge (B_1) to be true, but (B_1*) to be false. Now the vacuist may point out that orthodox philosophical practice leads us to conclude that both counterpossibles are vacuously true. But there is nothing to stop the particle physicist from pointing out that scientific practice leads us to conclude that one is true, and the other false. In short, the scientist need not be persuaded by what the vacuist has to say. Furthermore, if we are to base our judgments on the views of either, it seems we should base them on the views of the scientists regarding these scientific matters, rather than what the philosopher thinks about the truth/falsity of these statements.

Another place that Tan (2019) alleges science makes use of counterpossibles is in reasoning with idealised models. Science often treats planets as points for the purposes of performing calculations on their gravitational effect. Sometimes scientists also treat planes as if they are frictionless and liquids as if they are continuous. The use of such idealised modelling is prevalent throughout science, and once again arguably essential. For example, the sheer complexity of modelling a liquid as a series of discrete but bonded particles makes performing such calculations so difficult as to be unproductive, if not downright impossible. So scientists do tend to model things *as if they were these* idealised things. Tan (2019) alleges that these idealised things could not exist and could not fill the role of the substance being tested/investigated. For example, a continuous incompressible liquid could not do the things that water does, it could not be water. Yet we model water as if it were such an idealisation. Tan's claim is that we are modelling an impossible situation. Furthermore, reasonings based on such impossibilities constitute counterpossibles, e.g., "had water been a continuous incompressible medium…" (2019, p. 46). Such modelling is useful because the behaviour of water as it actually is closely approximates that of a continuous incompressible medium. The antecedent of this counterfactual model, i.e., "had water been a continuous, incompressible medium…" is metaphysically impossible. This makes the statement, as a whole, a counterpossible. Furthermore, it is a non-trivially true counterpossible.

We can explain why this statement is a counterpossible in similar ways to the diamond case. It is held that necessarily, water is identical to H_2O. As such, water has to be built up out of H_2O, and nothing that is made of anything else can be water. H_2O is not a continuous, incompressible medium, it is a series of bonded but discrete particles. So if something was such a strange medium, it would not be H_2O (and so not water). It would be metaphysically impossible for water to be a continuous, incompressible medium. But maybe people are not convinced here, again, perhaps they wish to define water by its theoretical role, rather than its chemical composition. Tan (2019, p. 46) thinks that even this view would lead to the statement in question being a counterpossible. One might allege that perhaps some continuous, incompressible medium can fulfil the role of water by acting exactly as actual water does. The problem is that this simply cannot be the case, a continuous, incompressible medium cannot fulfil the role of water. For example, a key property of water is that it is a solvent for particulate solids. No continuous, incompressible medium could ever act as a solvent for particulate solids, so no continuous, incompressible medium could ever fulfil the causal role of water (Tan, 2019, p. 46). Again, however we are defining water, it is metaphysically impossible that it be a continuous, incompressible medium. Yet we model it as such, so such models constitute counterpossibles.

One may be willing to accept this but deny that this counterpossible is non-vacuously true (or false). Tan's answer to this is to point to scientific practice and how things are actually done (and indeed how they have to be done). He alleges that such practices require us to treat these counterpossibles as non-trivially true. Tan uses the example of two competing models about the behaviour of water, M_1 and M_2. They both represent water as an idealised continuous fluid but they differ with respect to the viscosity they ascribe to water (2019, p. 47). To test these models, scientists will see how close the behaviour of water is to each model. Let us imagine they discover the predictions of one theory, M_1 to be very close to the behaviour of water, whilst the predictions of M_2 are further off. Scientists would rightly judge M_1 to be a true (or approximately so) theory, whilst M_2 would be false. Furthermore, they would take the following counterpossible to be false:

(C_1) "If water were a continuous, incompressible medium, then it would behave as M_2 predicts"

whilst taking this one to be true:

(C_2) "If water were a continuous, incompressible medium, then it would behave as M_1 predicts" (Tan, 2019, p. 47).

As we have already established, both are counterpossibles, and yet they have their truth values non-trivially. Orthodoxy might dictate that both of these are vacuous, but this does not constitute an argument for that being the case. Furthermore, the fact that it seems a worthwhile endeavour to reason using such counterpossibles is in fact evidence against the orthodoxy. If scientists were unable to reason so, then a large swath of scientific practice would disappear. Scientists need to use models like this and do so fruitfully, this would not be possible from vacuous counterpossibles, so we need to hold them to be non-trivial.

I think vacuists will struggle to respond to such cases from science. Scientific practice seems to require us to treat counterpossibles non-trivially, and this is important. The vacuist may have to say that scientists are simply mistaken, but this is unattractive as a position. Nor is it a position that scientists are likely to accept. If our logic/semantics conflicts with successful scientific practice then this seems to indicate a flaw in the logic/semantics rather than the scientific practice. Given this, non-vacuism may seem preferable. It will be helpful to consider one line of response the vacuist might make which I think fails. A vacuist could easily respond that indeed scientific practice does require us to treat some counterpossibles as non-trivial, but that this is not because such counterpossibles are non-trivial. Instead, perhaps what matters is that some scientific counterpossibles are assertable and some not, these are the ones we treat as non-trivial.

2.2. Assertability

A vacuist might say that the counterpossibles I want to describe as false are in fact merely not assertable (as discussed by Grice, 1975) and the ones I want to describe as non-vacuously true are assertable. This can be the case whilst all of them are true, and so I have not shown the vacuist thesis to be false, I have merely shown that some counterpossibles are assertable, and some are not. Perhaps, the class of "true" counterpossibles are assertable because they point to some underlying non-counterpossible truth, whereas the "false" counterpossibles fail to do this. For example, take the following pair of counterpossibles (Emery, Hill, 2017, p. 136):

(1a) If Obama had had different parents, he would have had different DNA.

(1b) If Obama had had different parents, he would have been two inches tall.

(1a) is assertable because it points to the underlying fact "(1c) Obama's parents were the cause of his having the DNA that he has" (Emery, Hill, 2017, p. 138). Whereas (1b) does not. Because they fail to do this, such counterpossibles are not assertable, and we mistake this intuition and say that they are false (Emery, Hill, 2017, pp. 137–138). However, as the orthodox view shows us, such intuition is mistaken, as all counterpossibles are true. The assertability of a statement, *s*, such as (1a) and the unassertability of its converse *s**, such as (1b), does not imply that *s* is true and *s** is false. The vacuist can then account for the views of non-vacuists whilst maintaining their theory.

This is an interesting point, but I do not think it threatens my view. Firstly, if it is the case that, for a given conflicting pair of counterfactuals, the assertability of one and the unassertability of the other does not imply that one is false, then it is also the case that it does not imply that they are both true. As Sendłak (2021) argues, the same pattern of assertability can be found in non-counterpossible counterfactuals, and whilst failing to imply that one is false, it also does not mean that both become true, for example as Sendłak says:

> [T]he assertion of "If Christopher Columbus had reached the place he was planning to reach in 1492, he would have arrived in India" can be explained by the fact that this allows one to indirectly express a more substantial proposition that is related to the asserted proposition in subject matter, e.g., "Christopher Columbus was planning to reach India". (2021, p. 11)

Whereas the converse "If Christopher Columbus had reached the place he was planning to reach in 1492, he would not have arrived in India", should intuitively be false, but under the Emery and Hill analysis, the truth value of the first sentence should not affect the truth value of the second, and so we could also view it as true. But crucially we can explain that the reason we intuitively think it is false is due to its unassertability. Sendłak claims that if we view this as problematic in the counterpossibles case, it is equally problematic in the counterfactual case, and that one could then hold a vacuist view of counterfactuals. Given the intuitive falsity of vacuism about counterfactuals, this is obviously a problem for a vacuist account that would endorse this (Sendłak, 2021, p. 11). Whilst it is true that a statement can fail to be assertable (for various reasons) without failing to be false, it does not mean that each and every statement which fails to be assertable also fails to be false. Emery and Hill (2017) try to introduce a gap between the unassertability of something and its falsity, the problem in the way they do this is that it creates a total disconnect between assertability/unassertability and the truth of a statement, in doing so they miss the target they aim for.

As noted, the result that science seems to rely on non-trivial counterpossibles is significant. Moreover, it is arguably a result we should favour over the traditional semantics. If scientists need to treat counterpossibles as non-trivial, then our accounts of counterpossibles need to treat them as non-trivial. One way in which a defender of vacuism might respond is to say that scientists do not need to treat counterpossibles as non-trivial, instead treating them as trivial but assertable/unassertable. This would not work though, the kind of arguments used for this could also be used to show that ordinary counterfactuals are trivial. This is clearly false, so something must be faulty with the argumentation. This way of saying that counterpossibles are merely assertable/unassertable will not work.

At this stage, the most we can have shown is that at least some counterpossibles are non-trivial, plausibly a large class of scientific ones. This of course does not show that all counterpossibles are non-trivial. As we noted at the start, on the face of it there might seem to be a difficulty with non-vacuous countermathematicals. Given that we need all counterpossibles in proofs by reductio to be true, the vacuist seems to be in a strong position. I think we can extend the spirit of why scientific counterpossibles are non-trivial to the case of countermathematicals and show that there are also non-trivial examples. First, it will be worth going over Williamson's (2018) discussion of why countermathematicals should be vacuous, as it will highlight some important points.

3. Counterpossibles in Mathematics

3.1. Williamson's Case

Williamson (2018) discusses the use of counterpossibles in mathematical proofs using reductio ad absurdum. As a hallmark example of this, he uses the proof that there is no largest prime number, known as Euclid's theorem. Williamson stresses that one does not necessarily need to phrase mathematical proofs in terms of counterfactual conditionals, but that it is a legitimate and natural way of doing so. So regardless of particular views on counterpossibles, all parties need an explanation of why this reasoning is legitimate and works. Williamson borrows the example from Lewis (1973, p. 25):

(L) If there were a largest prime, p, $p! + 1$ would be prime.

(M) If there were a largest prime, p, $p! + 1$ would be composite.

Williamson (2018, p. 363) helpfully summarises this proof: of (L) he explains that it holds because "if p were the largest prime, $p!$ would be divisible by all primes (since it is divisible by all natural numbers from 1 to p), so $p! + 1$ would be divisible by none" (2018, p. 363). Of (M) he points out that it holds because "$p! + 1$ is larger than p, and so would be composite if p were the largest prime" (Williamson, 2018, p. 363). Given that both these conditionals have the same antecedent, we are entitled to conjoin their consequents, resulting in:

(N) If there were a largest prime p, $p! + 1$ would be both prime and composite.

Given that the consequent of this counterfactual is a contradiction, we can deny the antecedent, and so say that in fact there is no largest prime. Quite obviously these are counterpossibles as well, because there cannot be a largest prime, that is a mathematical impossibility. Williamson and other vacuists, along with non-vacuists, will accept this as a good mathematical proof. In other words, everyone should accept all of (L)–(N) as true. Williamson's strategy is then to offer another proof by contradiction, using vacuous counterpossibles, which he says vacuists can accept easily, but that non-vacuists cannot accept, and cannot reject without rejecting Euclid's theorem. If non-vacuists deny the truth of the premises in Williamson's proof, he alleges they must also deny the truth of the premises in Euclid's theorem. Since rejecting such a proof would be unacceptable, we have a strong reason to doubt non-vacuism; so Williamson's argument goes. Before explaining why I do not think this argument works, I will spell out Williamson's second proof.

Williamson asks us to consider someone who answered "11" to "What is 5 + 7?" but who mistakenly believes that they answered "13", and utters the following counterpossibles, for the non-vacuist, (O) is false, whilst (P) is true (2007, p. 172):

(O) If 5 + 7 were 13, I would have got that sum right.

(P) If 5 + 7 were 13, I would have got that sum wrong.

Williamson is not persuaded by the initial intuitiveness of such examples:

> [T]hey tend to fall apart when thought through. For example, if 5 + 7 were 13 then 5 + 6 would be 12, and so (by another eleven steps) 0 would be 1, so if the number of right answers I gave were 0, the number of right answers I gave would be 1. (2007, p. 172)

If the number of right answers the person gives is 0, i.e., they give a wrong answer, then the number of right answers they give is 1, i.e., they get the sum right. So both counterpossibles are going to turn out to be true. Williamson then asserts that this is a result that the vacuist can get and accept, but that the non-vacuist cannot. He claims this points in favour of vacuism about counterpossibles. However, there is room for debate here. In particular, Brogaard and Salerno develop a series of objections against Williamson's reasoning.

3.2. Brogaard and Salerno's Objection

Brogaard and Salerno (2013) analyse Williamson's argument a bit more in depth and draw out the extra steps Williamson himself alludes to. The conclusion Williamson draws is that "if the number of right answers I gave were 0, the number of right answers I gave would be 1", hence, both (O) and (P) are true. The steps that Williamson abbreviates will be something akin to, if not exactly the following (Brogaard, Salerno, 2013, p. 649):

(i) If 5 + 7 were 13, then 5 + 6 would be 12.

(ii) If 5 + 7 were 13, then 5 + 5 would be 11.

…

(xi) If 5 + 7 were 13, then 5 + – 4 would be 2.

(xii) If 5 + 7 were 13, then 5 + – 5 would be 1.

It seems to be that what Williamson's argument is, at this point, is that worlds in which 5 + – 5 = 1 are also worlds in which 0 = 1, because we can substitute 5 + – 5 for 0. So we can conclude that:

(xiii) If 5 + 7 were 13, then 0 would be 1.

And so we get to Williamson's (2007, p. 172) conclusion that "if the number of right answers I gave were 0, the number of right answers I gave would be 1", with (O) and (P) both being true. Brogaard and Salerno go on to object that we can reject Williamson's proof here because he does not do a good enough job in establishing that the closest impossible world in which 5 + 7 = 13 is also one in which 5 + 6 = 12 (2013, p. 650). At this stage, we can return to Williamson's (2018) argument against non-vacuism.

The charge is that if non-vacuists reject Williamson's proof on the grounds that we have not established that the described world is the closest impossible world, then they must also reject Euclid's theorem for the same reason. Mathematicians will not concern themselves with the relative closeness of impossible worlds when producing proofs by contradiction, they will just produce the proof. So there is no evidence that the closest impossible world in which there is a largest prime, *p*, is also a world in which *p*! + 1 is both prime and composite (Williamson, 2018, p. 363). Non-vacuists will then be compelled to either reject Euclid's theorem, or to find a way of showing that the closest impossible world in the prime number case is indeed the world that Euclid's theorem describes. However, there is of course no guarantee that the same process cannot be performed for Williamson's proof, which would seem to tell against the non-vacuist. Essentially then, we should be viewing both counterpossibles in both cases as true, this is exactly as the vacuist describes and expects, but not as the non-vacuist does (Williamson, 2018, pp. 363–364). Having seen Williamson's argument we are in a position to respond to it. I think, at this stage, it will be worth making some clarifications about vacuism, and what Williamson has established so far, and also to build upon Brogaard and Salerno's objection, because whilst it might not work in its current form, I think there is an important idea contained within it.

Williamson claims that the counterpossibles used in Euclid's theorem and in his own proof are all true, because they follow from mathematical reasoning. The vacuist can obviously account for this, but the non-vacuist cannot, so Williamson claims. Perhaps the non-vacuist intuition that, for example, (O) and (P) have different truth values stems from some commitment that for any pair of counterfactuals that have contradictory consequents, but the same antecedent, at least one must be false. Williamson will claim that this failure to deliver the verdict of mathematics is a significant drawback of the non-vacuist account, and so we should reject such an account. I think non-vacuists can respond to this though. Not only has Williamson failed to successfully establish vacuism, I do not think that these mathematical proofs even constitute an argument for it.

3.3. The Problem With Vacuism

One could say that non-vacuists do not have to reject Williamson's proof. Certainly, Brogaard and Salerno did so under the banner of non-vacuism, but this is not an inherent commitment of that theory. There is nothing inherent in non-vacuism that says one cannot accept Williamson's proof. Perhaps Williamson has shown that all those counterpossibles are true, but that does not mean he has shown that vacuism is true, or even that all counterpossibles are true. Vacuism is essentially the thesis that all counterpossibles are vacuously true, because their antecedents are necessarily false.[4] The truth of the counterpossibles comes from this fact, this is what makes the counterpossible true. The problem is that vacuism plays no role in making (L), (M) or (N) true in Euclid's theorem. As Williamson himself says, they are true because they are mathematical results; "[(L)–(N)] should be true, for they are soundly based on valid mathematical reasoning" (2018, p. 363). But this is independent of vacuism. Williamson correctly points out that a semantic theory needs to produce this result, and indeed vacuism does, but for one it is unclear that it does so for the correct reasons, and two, it is not the only semantic theory that does this. The truth of (L)–(N) is a mathematical result, they are true for reasons stronger than the mere impossibility of the antecedent. Compare this with:

(L*) "If there were a largest prime p, $p! + 1$ would be a set", or

(L**) "If there were a largest prime p, $p! + 1$ would be an infinite set".

I think mathematicians would want to reject these conclusions, they would want to say that these statements were false, as would non-vacuists. They would be false because they would be based on faulty mathematical reasoning. However, on Williamson's account, they would come out as true. Consider a world in which mathematical practice was systematically wrong. For whatever reason, mathematicians just get the wrong verdict when talking about these matters. In such a world, clearly some counterpossibles would be described as false by the mathematicians, but they would all be described as true by the vacuist. The result from vacuism and the result from mathematical practice are distinct results. I think this in itself constitutes a criticism of vacuism. We have already discussed cases in science that seem to require non-trivially true/false counterpossibles, so it seems vacuism about all counterpossibles might be false. But restricting vacuism to mathematical counterpossibles is a redundant thesis, this amounts to a claim that all the mathematically proven statements are true. Or, if mathematical practice told us that a particular counterpossible was false, it would amount to disagreement with mathematical practice. This second alternative is exactly what the vacuist charges the non-vacuist with as a significant problem, and yet it seems they might be vulnerable to exactly the same point. But the point against Williamson and the vacuists is not merely that his account produces the wrong results in certain cases, that would merely be a reframing of the intuitive arguments for non-vacuity. Instead, the point is that in the mathematical cases he would appeal to, although he gets the right result, the result is obtained regard-

[4] Non-vacuism of course being the thesis that there are at least some non-trivial counterpossibles.

less of his theory. We can see this by the fact that non-vacuists accept the result that both statements are true in the case of Euclid's theorem, and they do so on non-vacuist grounds; because it is a mathematical result. Williamson's mistake comes from the fact that he assumes that, to take the counterpossibles he makes in his proof as being true, the non-vacuist would have to subscribe to some form of vacuism; but this is not the case. One can take (L)–(N) to be true without being a vacuist,[5] and that is so because, as Williamson points out, they follow from mathematical reasoning. Our intuitions led us to think that (O) and (P) had different truth values, but mathematical reasoning showed us this was wrong. That is something the non-vacuist can accept, just because non-vacuism is committed to some counterpossibles being non-trivial, it does not mean that on each occasion that our intuition points to counterpossibles having different truth values, we are right. Importantly again, the mathematical counterpossibles we have discussed are not even trivially true. They follow from mathematical reasoning so they are substantially true.

We have seen how Williamson's proof works and how the non-vacuist can equally accept this result. Williamson's proof does seem to fall out of standard mathematical definitions of addition, the successor principle, etc. But another point to be considered is whether or not Williamson has genuinely evaluated the truth value of the counterpossible in the way it should be. One important point to discuss is the Baron, Colyvan and Ripley (2017) discussion that Williamson's proof fails to consider the closest counterpossible scenario. But first it will help to consider an important distinction that I think is very relevant to the current topic, the distinction between genuinely conceiving of a distinct world, and considering a conjecture at the actual world.

3.4. How to Genuinely Consider a Distinct World

I think Brogaard and Salerno (2016) have captured something with their objection. They charge Williamson with not conceiving of the closest possible world. Williamson says that rejecting his proof on these grounds would mean we also have to reject any mathematical proof by contradiction, such as Euclid's theorem. This is clearly unattractive, and so we should not reject his account. But I think that this objection has targeted something important, albeit in the wrong way. Williamson's proof does not work by describing the closest world (in which the conjecture is true) to the actual world, but this is because his proof does not consider a distinct world at all. What Williamson has done is show that the actual world cannot be a particular way, given what we already know. This is a point worth spelling out in some detail.

Let us consider the two different kinds of process we might engage in using counterfactuals that were mentioned in the introduction. In the first case, the

[5] Indeed, it seems that one can understand and reach this result, without any view on the vacuity/non-vacuity of counterpossibles.

truth value of a statement/hypothesis might be unknown, and so we want to find out/demonstrate whether it is true or false. To do this, we use the hypothesis to derive a prediction and make a counterfactual using the hypothesis as the antecedent and the prediction as the consequent. If the prediction turns out not to be the case, we can use this to show that the antecedent was false. This is what we are doing in the example of Bohr's theory and in Williamson's proof. We say if one thing were the case, a second thing would also be the case, as the second is not the case, we can say that neither is the first. If 5 + 6 were 13, then 0 would be 1, 0 is not 1, so 5 + 6 is not 13. Now as it happens, in both these cases, the antecedents turn out to be necessarily false, and so the counterfactuals involving them are actually counterpossibles. The vacuist says that as counterpossibles are trivially true, these particular ones are trivially true. However, these particular counterpossibles are useful. The counterpossibles that non-vacuists wish to call true, (B_1 and xiii) contain consequents that contradict our experience, as such these are the ones which can actually be used to show the antecedent to be false. This is the process one might engage in to show that the antecedent of a counterfactual is false, and this is the process that Williamson is engaging in. However, there are situations where we already know the truth value of the antecedent, and these are the cases I want to focus on.

There may be cases when we know that a statement is false, perhaps even necessarily false, but we want, for whatever reason, to explore what would be the case if in fact it were true.[6] This is what we are doing in the case of modelling water as a continuous medium and in the Brogaard and Salerno example. In these cases, we know that the antecedent is false, we know that water is not an incompressible, continuous medium, and we know that 5 + 6 is not 13, but we want to find out what would be the case if they were. In order to find out what would be the case if they were true, we have to assume them to be true. To do that, we need to sacrifice some assumptions to avoid contradictions, e.g., that water is not a continuous medium and that 5 + 6 is not 13. Doing this would prevent us running into contradictions and so the counterpossible would not be trivial, because we could produce a false counterpossible by making a false statement about what would be the case if the impossible antecedent were the case. It is worth pointing out that we already make the distinction between these kinds of projects in the case of ordinary counterfactuals. Let us take the case of a crime scene investigation; in conjecturing how the murder victim was killed, the detective will make hypotheses. Perhaps one of these hypotheses is that the victim was shot. The detective may then form a counterfactual of the form "if it were the case that the victim was shot, there would be a gunshot wound on the body". If no gunshot wound is found, the detective can conclude that the ante-

[6]A similar idea to what follows occurs in Sendłak's (2021, pp. 16–18). However, this idea presents an important critique of vacuism concerning counterpossibles and so I think it warrants more attention and exploration than it has been given elsewhere.

cedent was false. In such a case, counterfactual reasoning has been used to discover that something is false.

Alternatively, sometimes we know that a statement is false, but we want to work out what would be the case were it true. If you cycle to work and your tyre bursts, resulting in you being late to work, you can usefully say “if I had driven to work, I would not have been late”. We know the antecedent is false, but we assume it to be true, and reject assumptions like you actually having ridden your bike in order to make non-trivial statements. If we did not reject assumptions, we would simply run into contradictions and end up proving that you had in fact cycled to work, but this is not what we wanted to do. This distinction between kinds of reasoning is present in the case of ordinary counterfactuals and it is not clear why it should not be present in the case of counterpossibles. With this distinction more clearly in mind, we can assess Williamson’s account of counterpossibles. I think we can diagnose why Williamson thinks he has got the result he does, whilst also explaining Brogaard and Salerno’s objection. Put simply, Williamson is engaged in the first kind of reasoning process mentioned above, whilst Brogaard and Salerno are engaged in the second.

Williamson’s proof is simply a proof that $5 + 7 \neq 13$. That is a perfectly legitimate thing to do and might be useful in some circumstances. But the reason that proof works, is the same reason the Euclid proof works. It works because we hold fixed everything we know about the world (in this case mathematics), and then show that given that, a particular fact could not be the case. In Euclid’s proof, we hold fixed facts about prime numbers, where in the number sequence they tend to appear for example. We then want to show that the assumption that there is a largest prime number is inconsistent with this. In doing this, we have not considered a different world, we have not moved from our world. Because we are showing that something cannot be the case, *at our world*. In Williamson’s proof, he has perhaps held fixed facts about addition, the successor principle, etc. and then shown that given these things, $5 + 7 \neq 13$. But note, this is not to consider a world in which $5 + 7$ is 13. Because if we are considering a world in which $5 + 7 = 13$, this cannot be a world in which it is also the case that $5 + 7 \neq 13$. Williamson has not considered a different world, he has considered the actual world and shown that a certain statement is false here. Now, all the statements Williamson invokes might be true, but once again, they would be true non-vacuously, because they would be mathematical results. But it is not clear that he is genuinely considering a counterpossible.

Williamson (2020, p. 18) describes a process he calls the Suppositional Procedure (SP). In order to assess the truth of a conditional if A then C, one has to suppose that A and then judge whether, on the basis of that, it is also the case that C. Importantly, this simple form of the SP makes no mention of the possibility of A or C, simply that one must suppose A. One intuitive claim about supposing is that we have to suspend our disbelief in some way, perhaps just as in the case of make-believe games. Leng (2010) talks about make-believe in mathematics and describes the process as representing real objects in some way. Specifically it is to

> [i]magine of real objects that they are other than they really are. It is clear in these cases that we are sometimes being required to imagine something *false* concerning the nature of such objects: we know that the tree stumps are not *really* bears; that the fluids are not *really* continuous. (Leng, 2010, p. 159, author's emphasis)

If we know *A* to be false, but want to suppose it for some purpose, we have to reject other facts which would rule *A* out. The move I wish to make should be clear now, in Williamson's proof above, he has simply failed to suppose[7] that 5 + 7 = 13. Let us consider a more in depth spelling out of a true suppositional process.

Take any proposition, *P*, if one is to consider a world at which it is the case that *P*, then the world considered must also be a world in which ~~*P*. Now this is not to say that there cannot be worlds which contain contradictions. If we are considering a world in which it is raining and not raining (same place, same time), it seems like we are considering a world in which *P* and ~*P* (neglecting to include ~~*P*). But this misses the mark a little bit. We are considering a world in which it is the case that it is raining and it is not raining. This is a proposition, *Q*. If we need to consider that world, then we also need to be sure that it is a world at which ~{~[it is raining and it is not raining]}, i.e., that it is also a world at which ~~*Q*. Williamson fails to consider a world at which 5 + 7 = 13, because he does not ensure that it is also a world at which it is the case that ~{~[5 + 7 = 13]}. And holding fixed the background mathematical facts, just as in the Euclid case, is key to the proof working, because the proof aims at showing that *at the actual world*, something is not the case. It is worth noting as well, that is not some method peculiar to counterpossibles. This is exactly the process we need to engage in for ordinary counterfactual scenarios.

Let us take the straightforward counterfactual "If Julius Caesar were alive today then…". We have a number of assumptions that we are committed to at this world, the average lifespan of a human being currently sits at around 81 in the UK. Perhaps given this, we also assume that anyone who was alive at the time of Julius Caesar is now dead, including Caesar himself, i.e., we assume that ~[Julius Caesar is alive today]. In order to genuinely consider a world at which it is the case that Caesar is alive today, we need to reject this implicit assumption for the purposes of conceiving. We need to explicitly make sure it is a world at which ~{~[Julius Caesar is alive today]}. If we do not do this, then we will of course run into inconsistencies, and potentially end up proving that our conjecture (that Caesar is alive) is incorrect. But this is not to genuinely conceive of a distinct world, this is a different process. If we were to consider the closest world in which 5 + 7 = 13, then we are going to have to jettison some mathematical assumptions. In doing so, it is not clear that all of Williamson's statements would follow mathematically, and so be true. In fact, it actually seems more likely that Williamson's proof will not go through, some statements will come out false. In just the same way as if we genuinely considered a world at which

[7] A more in depth discussion of why I assert this takes place in Section 4.3.

there was a largest prime, likely Euclid's theorem would not work. But this should not be surprising, a world with a largest prime is a world where Euclid's theorem is false.[8] This does not threaten mathematical practice, because this is not the aim of mathematical practice.

There are of course limits to how far this process can go, both in terms of unavoidable contradictions and in terms of the considered scenario being so distant from our own as to be irrelevant. But such things can be assessed on a case by case basis, Baron et al. (2017) propose a method in this style for "chasing out" contradictions from the immediately relevant vicinity of the counterpossible scenarios, in some cases the relevant vicinity will be much larger than in others, but the process is the same.[9] One may be concerned that such a process will in fact have no end, and that as we are dealing with metaphysical necessity, there will always be contradictions in the counterpossible scenario we imagine. Alternatively, the concern may be that the process takes so long that in rejecting background assumptions we end up with a completely different arithmetic system in which everything works so differently that we cannot retrieve any useful conclusions from consideration of the scenario. Baron et al. (2017, p. 8) address such concerns by pointing out that a similar process occurs in the consideration of ordinary counterfactuals.

In ordinary counterfactuals, we may run into contradictions in considering the scenario, but we simply reject all and only those relevant for whatever our purposes may be. For example, in considering the case of whether Suzy's throwing of the rock caused the window to break, we may consider counterfactuals beginning "If Suzy had not thrown the rock…". In such cases there are of course inconsistencies, in the scenario we are considering it may be the case that Suzy indeed moved to throw the rock but that the rock did not move for some unspecified reason. Or it could even be that Suzy made the decision to move her arm but that it simply did not happen (Baron et al., 2017, p. 8). It simply is not the case that we go back through the entirety of history to make this scenario consistent. In fact we tend to ignore the inconsistencies and just conceptualise Suzy failing to throw the rock, without necessarily filling in the background details as to how

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[8] Berto et al. (2018, p. 704). discuss a similar point related to Euclid's theorem. In the context of a reductio proof we should hold everything fixed, but in other contexts it might make sense to jettison some assumptions and in such cases not all statements would mathematically follow (i.e., some counterpossibles would be false).

[9] One concern I have with the specific way Baron et al. (2017) go about the process in their paper is that it seems they might in fact no longer be considering counterpossibles because they redefine what various mathematical operators mean, specifically addition. It seems at that stage that rather than considering impossible ways for the specific mathematical system we have to be, they might simply be considering a different mathematical system, and so this simply seems like a counterfactual. Compare this to a counterfactual like "Had the queen in chess not been able to move diagonally, then…" this does not seem to be a claim about the specific set of rules we have for chess currently, but rather about a different set of rules.

that failure was realised. It seems that the same process should take place in the countermathematicals case. Baron et al. (2017, p. 9) think that when we dispense with the immediate contradictions in the mathematical case we can leave it there and ignore the rest, even if actually addressing all the contradictions would be an infinite process. Now of course it might be the case that addressing the immediate contradictions in an ordinary counterfactual case is much simpler and a much smaller job than addressing the immediate contradictions in a mathematical counterpossible. But there is no reason to think that this process is anything more than a difference in degree. If we perform this process then we can consider internally consistent (but impossible) scenarios and try to determine what would/would not be the case, were these scenarios to take place.

3.5. Countermathematicals in Explanation

So far we have discussed why it is that we should judge scientific counterpossibles to be non-trivial. We have also shown how there are different uses of counterpossibles depending on which sort of reasoning we are engaged in (either discovering the truth value or reasoning on the supposition of truth regardless of the actual truth value). It is time to extend this to the mathematical case. We have already seen how counterpossibles play a role in the first kind of reasoning. When we aim to test a mathematical hypothesis we hold everything else fixed and see if we run into contradictions. If we do, then the antecedent is false. In such cases, it might turn out that all the countermathematicals involved are true. But importantly, they are not true because vacuism is correct, they are true because they follow from mathematical reasoning. Euclid's theorem discussed earlier was one example of this. It also seems plausible that Williamson's proof (2007; 2018), is an example of this kind of counterfactual reasoning. But countermathematicals can also be used in the second kind of reasoning process, to explain something in the world.

There are many examples of this, but a key one is the discussion by Lange (2017) about distinctively mathematical explanations.[10] Although this work of Lange's does not enter into these areas of counterpossible debate, I think it does bear upon it in a number of ways. One (very simple) example of a distinctively mathematical explanation would be something akin to "The reason that Jane cannot divide her 23 strawberries equally between her 3 children (without cutting), is because 23 is indivisible by 3". In context of the scientific explanations we considered earlier, this is quite similar to the explanation of why diamond does not conduct electricity. So, to put the mathematical explanation in counterfactual terms (as is legitimate practice) we can say "Had 23 been evenly divisible

[10] This kind of explanation is parallel to the usage of counterpossibles to explain the poor conductivity of diamond and the movement of water as described in Section 2.1. We might know that the antecedent is false, but we want to suppose it to be true to highlight some sort of dependence relation.

by 3, then Jane would have been able to divide her 23 strawberries evenly between her 3 children (without cutting)". In the case of a counterfactual like this, we are not trying to discover the truth value of the antecedent. We know it is false, indeed we know it is impossible. What we are trying to do is work out what would happen if it were true. We have to suppose the antecedent to be true. In order to suppose it to be true, we simply cannot hold everything else fixed. When we start to jettison assumptions (for starters, we might get rid of the fact that 23 is prime), we will no longer run into a straightforward contradiction between the antecedent and consequent. Yli-Vakkuri and Hawthorne remark when discussing provability in mathematics that "[…] '$\vdash$' expresses provability in mathematics—by which we mean pure mathematics. $\Gamma \vdash A$ only if both A and all of the statements in Γ are pure mathematical statements" (2020, p. 560). When we are discussing counterpossibles which contain a non-mathematical consequent, the consequent will not follow mathematically from the antecedent. As such, the counterpossible as a whole may well turn out to be false. The mistake of the vacuist is in thinking that the first kind of reasoning process is the only one, or that it is the most important one. If it is the case that all the countermathematicals used in the first kind of process are true, it is not because of vacuism, it is because of mathematical practice and its results. In the second case, it is simply not the case that they all turn out true, their truth value will vary from world to world, just as with counterfactuals.

4. Potential Problems

4.1. Do Mathematicians Use Counterfactuals?

One general point to bring up is whether or not mathematics does indeed use counterfactuals, as opposed to merely appearing to use them through language choice but actually relying on something else.[11] Non-vacuists about countermathematicals clearly think that mathematics makes use of them. But it is important to point out that many prominent vacuists also think this. For example, as Yli-Vakkuri and Hawthorne say, "we will argue, mathematics makes use of the counterfactual conditional…" and that this usage "is by no means a marginal feature of mathematical discourse" (2020, p. 552). Indeed they themselves ultimately view it as indispensable. Perhaps the most vocal vacuist, Timothy Williamson, also concedes that we must account for the use of counterfactuals in mathematics as it is a legitimate practice (2018, p. 363). Reutlinger et al. (2020) began a more formal study of mathematical language and, from those preliminary results,[12] it seems to be the case that mathematicians frequently use counterfactuals. Now of course, one could maintain a commitment to this choice of language being a facade, perhaps disguising material conditionals. However,

[11] Thank you to a reviewer for bringing up the importance of clarifying this point.
[12] Available in Section 5 of that paper.

given the prevalence of seeming-counterfactuals in mathematics, and given that mathematicians seem to be taking themselves to be talking in counterfactual terms, this would be quite a revisionary view of mathematical practice. As such, I think it would require extensive independent justification to be considered as a serious objection. Whilst both vacuists and non-vacuists seem to be taking counterfactual usage for granted, I think we can simply assume the usage is genuine for the purposes of this debate.

4.2. How Do Mathematicians Use Countermathematicals?

Even granted that mathematicians genuinely appeal to countermathematicals in their writings, it is unclear how they are appealing to them, i.e., if they are appealing to them as vacuous or not. Williamson would clearly disagree with my claims that the judgements of mathematicians about specific countermathematicals would match the non-vacuist judgement. It is worth discussing some evidence in favour of the non-vacuist view. Yli-Vakkuri and Hawthorne (2020, p. 567) say that, in conversations with mathematicians, they will tend to assert counterpossibles like the following:

(TB): "If AC were false, then the Tarski-Banach theorem would not be provable from the truths of set theory"

whilst denying counterpossibles like:

(TB)^1: "If AC were false, then the Tarski-Banach theorem would be provable from the truths of set theory".

I think this is exactly as the non-vacuist should accept (and indeed as I assert), and confusing only for the vacuist. The reason for this is that (TB) is true because the consequent would follow if the antecedent were true. Part of what is for AC to be false is for the Tarski-Banach theorem to fail to be provable from the truths of set theory.[13] Thus, (TB)^1 is false because if the axiom of choice were false, it would not be possible to prove the Tarski-Banach theorem from the truths of set theory, such a proof requires the truth of the axiom of choice. This element of mathematical practice is an anomaly for the vacuist, as noted by Yli-

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[13] A good summary of this idea is available in Sendłak's (2021). Sendłak argues that counterpossibles such as "Had paraconsistent logic been true at the actual world then…" are paraphrases of statements like "According to the story of paraconsistent logic…". If the consequent in the paraphrase makes the statement false overall, then the counterpossible equivalent should also be false. By Sendłak's argument, Williamson (2020, pp. 129–130) is simply wrong when he accepts "...if the Bible is to be believed, there are angels" and also accepts "if the Bible is to be believed, there are no angels". Believing the antecedent to be false does not justify accepting the second statement as true, because that is simply false according to the story of the Bible.

Vakkuri and Hawthorne (2020, pp. 567–568). This practice also extends to logicians discussing counterlogicals (counterfactuals with a logically impossible antecedent). This practice which seems to contradict vacuism is a problem for vacuists to solve. If this practice is stable then vacuists will have to be quite radically revisionary about mathematical/logical practice, an obvious weakness. Non-vacuists, however, have a prima facie explanation of this phenomenon; the reason that mathematicians deny such counterpossibles is because such counterpossibles are false. In these cases, the consequent does not follow from the relevant antecedent.

Further support for the non-triviality of countermathematicals can be found in (Jenny, 2018). Jenny proposes that mathematical practice implicitly relies on the assumption that countermathematicals are non-trivial, specifically in the case of relative computability theory. This is important work. Jenny also proposes (2018, p. 552) a project going forward whereby non-vacuists should aim to find counterpossibles in other areas, such as the sciences, to defeat vacuism on multiple fronts. As Jenny says

> Once we have a clearer picture of the areas where non-vacuous counterpossibles are indispensable and once we have model theories for these various classes of counterpossibles, we may then investigate to what extent we can integrate these model theories to come up with a unified and fully general theory of non-vacuous counterpossibles. (Jenny, 2018, pp. 552–553)

My paper can then be seen as a continuation of the Jenny project, an attempt to bring counterpossibles in these distinct areas together. This is also where my paper goes further than Jenny. This paper aims not merely to show individual cases of non-trivial counterpossibles in distinct areas, but also to show why these are non-trivial. I aim to show the process we need to engage in to get the result of non-triviality, along with the fully general theory that Jenny is looking for. I think the only way for non-vacuists to make a start on this general theory is to highlight the mistakes that vacuists make by making clear the requirements to genuinely conceive of something, and show how vacuists fail to do this.

4.3. Is Williamson in Fact Genuinely Conceiving of a Distinct World?

One may object to my criticism that Williamson (2018) has not considered a distinct world, and has simply considered the actual world. One way to do this can be drawn out from the work of Yli-Vakkuri and Hawthorne (2020). Yli-Vakkuri and Hawthorne say take a standard proof by reductio in maths, e.g., Euclid's theorem. In this proof, one initially supposes that there is indeed a largest prime. Then, given this claim and other established truths, they deduce various other statements and eventually show that the hypothesis in question was false. The allegation would be that I have unfairly characterised the mathematical process, because the above describes a situation in which one does suppose the false hypothesis to be true. This is not quite right though, and in fact we can use more discussion from the Yli-Vakkuri and Hawthorne paper to explain this. In their paper, they make the distinction between a consensus and a non-consensus context. As they say:

> In a consensus context the relevant axioms are taken for granted, it is common ground that they are being taken for granted, and no one is interested in challenging any of the axioms or in exploring the ramifications of giving up some but not all of the axioms […]. In a non-consensus context one is not entitled to assume that all of the axioms are true and hence also not entitled to assume that they are provable, since provability entails truth. (Yli-Vakkuri, Hawthorne, 2020, p. 566)

What I think it takes to genuinely suppose a statement/hypothesis, is to be in a non-consensus context. For it is only in a non-consensus context that you drop the assumptions you have that will immediately contradict the hypothesis. In a consensus context, the countermathematicals may all turn out to be true, but once again not vacuously so, because they followed from the relevant mathematics. In a non-consensus context, this is not the case. When one jettisons assumptions, one will not immediately run into contradictions, so the truth value of the counterpossibles will be up for grabs. To decide whether or not Euclid's theorem is a case of a consensus/non-consensus context, let us reiterate what goes on in that example. As Yli-Vakkuri and Hawthorne (2020, p. 558) say, we take a set of assumed axioms, $\mathbf{\Gamma}$, e.g., the Peano axioms, and *A*, which is the claim that there is a largest prime, and ultimately conclude *B*, our desired contradiction which shows us that the claim, *A*, was false. We should be able to see that, in their own terms, this sounds like a consensus context because the set of assumptions, $\mathbf{\Gamma}$, have not been modified. This matters because $\mathbf{\Gamma}$ will either directly contain the proposition ~*A*, or ~*A* will be a logical consequence of $\mathbf{\Gamma}$. In this way, consensus contexts fail to be a genuine conception/supposition of *A* being the case, because they implicitly assume that ~*A* is the case.

To make clear the implications for Williamson's argument, my allegation is that Williamson stays within a consensus context. This is insufficient for a genuine conception/supposition of *A*. In terms of the ways we might use counterfactuals discussed in the introduction, this is the first kind of reasoning process, not the second. It is a consensus rather than non-consensus context; and as I have claimed, the second kind of process is the one which can produce false counterpossibles. There is further support for this later in the paper when Yli-Vakkuri and Hawthorne describe a fictional community of mathematicians, "[f]or example, if *A* is the claim that there is a largest prime number, the point, if any, of a Boxer's assertion of $A \mathrel{\Box\!\!\rightarrow} B$ will be to contribute to an explanation of why there is no largest prime number" (2020, p. 566). In order to show that *A* is not the case, they have to keep in place the assumptions that will contradict it. Plainly this will be a consensus context which fails to be genuinely conceiving of a situation in which *A* is the case.

4.4. Is Counterpossible Usage a Fringe Phenomenon?

One of Williamson's key arguments in favour of vacuism is that counterpossibles are a fringe phenomenon. This seems to be implicit in his discussion in a number of places:

> [In a discussion of counterlogicals] it would be naive to take appearances uncritically at face value in a special case so marginal to normal use of language, for example by offering them as clear counterexamples to a proposed semantics of conditionals […] it is good methodological practice to concentrate on conditionals with less bizarre antecedents in determining our best semantic theory of conditionals […]. (2020, p. 60)

> After all, once the impossibility of a supposition is recognized, continuing to work out its implications is typically a waste of time and energy. (2020, p. 234)

> In linguistic practice, counterpossibles are a comparatively minor phenomenon, which is one reason why it is implausible to complicate the semantics of modalized conditionals in natural language just to achieve a desired outcome for them […]. (2020, p. 262)

However, I would simply deny that these are in fact fringe cases of counterfactuals. As we have seen, vast portions of scientific reasoning contain counterpossibles; mathematicians and logicians seem to use countermathematicals/counterlogicals respectively; and to engage in meaningful debate in metaphysics, it seems we might need to use countermetaphysicals. Given the wide usage of counterpossibles in all these domains, it makes little sense to describe these as fringe cases. Counterpossibles are a significant datum, and a semantic theory needs to account for their usage in a way that is not revisionary to the vast areas of practice which employ them. If, as Williamson says, such counterpossibles present a problem for a standard semantic theory, then I think that is simply a reason to reject that particular semantic theory, rather than be revisionary to all this practice.

5. Conclusion

One straightforward and orthodox reading of counterpossibles implies that they are all trivially true. However, this conflicts with a lot of intuitions we might hold. Of course intuitions only take us so far because not everyone holds them. But there is also strong precedent in the sciences to treat counterpossibles non-trivially. One reason to do this is that it seems that in cases of non-trivially true counterpossibles, we can reason from the antecedent to the consequent in some way. In non-trivially false counterpossibles, the consequent does not follow in this way. When we reject the assumption that the antecedent is false, we can use counterpossible form to discover the counterfactual dependence at play. For example, that the microphysical structure of diamond is responsible for its poor electrical conductivity, or to reason about what would have been the case if something impossible was the case, e.g., if Bohr's theory of the atom had been correct, we would have observed electrons in such-and-such a way. This reasoning can go wrong when we make a mis-ascription as to what would have been the case, resulting in non-trivially false counterpossibles. Despite apparent surface level difficulties, we can also extend the same reasoning process to intuitively non-trivially true countermathematicals. This also gives us space to have non-trivially false countermathematicals, when this reasoning process goes wrong. To engage in this kind of reasoning in either case we may need to, on some level, genuinely conceive of an impossible world. To consider a counterpossible, $A_i > B$, we have to genuinely conceive of a world in which A_i is the case, in doing so we have to reject our assumptions to the contrary. When we do this, some counterpossibles will turn out true, and some will turn out false. In other words, vacuism about counterpossibles is false.

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