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Article

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ON MARTIN-LÖF'S CONSTRUCTIVE OPTIMISM

SUMMARY: In his 1951 Gibbs Memorial Lecture, Kurt Gödel put forth his famous disjunction that either the power of the mind outstrips that of any machine or there are absolutely unsolvable problems. The view that there are no absolutely unsolvable problems is optimism, the view that there are such problems is pessimism. In his 1995—and, revised in 2013—*Verificationism Then and Now*, Per Martin-Löf presents an illustrative argument for a constructivist form of optimism. In response to that argument, Solomon Feferman points out that Martin-Löf's reasoning relies upon constructive understandings of key philosophical notions. In the vein of Feferman's analysis, one might be object to Martin-Löf's argument for either its reliance upon constructivist (as opposed to classical) considerations, or for its appeal to non-unproblematically mathematical premises. We argue that both of these responses fall short. On one hand, to be critical of Martin-Löf's reasoning for its constructiveness is to reject what would otherwise be a scientific advance on the basis of the assumption of constructivism's falsehood or implausibility, which is of course uncharitable at best. On the other hand, to object to the argument for its use of non-unproblematically mathematical premises is to assume that there is some philosophically neutral mathematics, which is implausible. Martin-Löf's argument relies upon his third law, the claim that from the impossibility of a proof of a proposition we can construct a proof of its negation. We close with a discussion of some ways in which this claim can be criticized from the constructive point of view. Specifically, we contend that Martin-Löf's third law is incompatible with what has been called "Poincaré's Principle of Epistemic Conservation", the thesis that genuine increase in mathematical knowledge requires subject-specific insight.[1]

KEYWORDS: optimism, pessimism, Martin-Löf, Gödel's disjunction

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Optimism and Pessimism

In his 1951 Gibbs Memorial Lecture *Some Basic Theorems on the Foundations of Mathematics and Their Implications*, Kurt Gödel argued for his famous disjunction: "Either […] the human mind (even within the realm of pure mathematics) infinitely surpasses the powers of any finite machine, or else there exist absolutely unsolvable diophantine problems of the type specified" (Gödel, 1951, p. 310).

While both disjuncts have received much attention (see, for example, Lucas, 1961; Penrose, 1989; 1994; Horsten & Welch, 2016), the former has been the primary focus of philosophical discussion surrounding Gödel's Disjunction. In this article, we focus on the latter. The view that there are no absolutely unsolvable diophantine problems is optimism. The alternative thesis is pessimism, that there exist absolutely unsolvable diophantine problems.

What does Gödel have in mind by an absolutely unsolvable diophantine problem? A diophantine equation is one in which only integer solutions and coefficients are used. A diophantine problem is the question of whether a given diophantine equation has a solution. Notably, one can verify if a given assignment to the variables is a solution in a finite number of steps. The question of whether or not a given equation has a solution we then understand as the disjunction of the claim that there is a solution to that equation and its negation. This, in turn, is the traditional way of understanding the law of excluded middle as the formalization of optimism, which we see in L.E.J. Brouwer, for example (Brouwer, 1908, p. 109).

The question of whether or not there exist absolutely unsolvable problems has its proximal roots in 19th century philosophy of science. Emil du Bois-Reymond famously closed his 1872 Berlin address, as translated by Andrea Reichenberger: "As regards the riddle of the nature of matter and force and how they are able to think, we must resign ourselves once and for all to the far more difficult verdict: 'Ignorabimus'" (Reichenberger, 2019, p. 53).

du Bois-Reymond's view here is that there are portions of reality that are inaccessible to us. While these are not the sort of mathematical problems we have been discussing, the view here is that there are some metaphorical "corners" that we will never be able to see around.

Opposite the pessimism of du Bois-Reymond we find the optimism of David Hilbert. In his 1900 address to the International Congress of Mathematicians in Paris, he claimed:

> This conviction of the solvability of every mathematical problem is a powerful incentive to the worker. We hear within us the perpetual call: There is the problem. Seek its solution. You can find it by pure reason, for in mathematics there is no 'ignorabimus'. (1902, p. 445)

On this view, the mathematical "worker" is motivated by the knowledge that their task is not some thankless Sisyphean task but rather one they can actually complete.

Martin-Löf's Constructive Optimism

In his 1995 and, revised in 2013, *Verificationism Then and Now*, Per Martin-Löf presents a case in favor of optimism. Making use of several laws for which he provides philosophical justification, he argues:

> [T]here are no absolutely undecidable propositions. And why does this follow from [the third law, the claim that if a proposition cannot be known to be true then it can be known to be false]? Well, suppose that we had a proposition which could neither be known to be true nor be known to be false. Then, in particular, it cannot be known to be true, so, by the third law, it can instead be known to be false. But that contradicts the assumption that the proposition could not be known to be false either. (2013, pp. 12–13)

Imagine that a given proposition is absolutely undecidable, which is just to say that the associated problem is unsolvable in the sense we used above. In terms of knowledge, given that it is in fact absolutely unsolvable, this means that it cannot be known to be true and it cannot be known to be false. But, if a proposition cannot be known to be true, then, Martin-Löf argues, it can be known to be false. This is in virtue of his third law. The thought is that if it is impossible that a is a proof of A for any a, then we can conclude a refutation of A. But, if we have a refutation of the proposition in question, then the problem is not absolutely unsolvable, which contradicts our original assumption. Therefore, there are no absolutely undecidable propositions. Call the above articulation of optimism constructive optimism.

There is a clear step worth examining in more detail, that from the impossibility of knowing the truth of the proposition we can move to the possibility of knowledge of its falsehood. This, however, will be the focus of the second half of this paper. Let us first turn to a different sort of objection to Martin-Löf's argument. Solomon Feferman, in "Are there Absolutely Unsolvable Problems? Gödel's Dichotomy", comments:

> Indeed, Per Martin-Löf has proved exactly that, in the form: There are no propositions which can neither be known to be true nor known to be false [...]. However, this is established on the basis of the constructive explanation of the notions of "proposition", "true", "false", and "can be known". (2006, p. 147)

Feferman continues:

> For the non-constructive mathematician, Martin-Löf's result would be translated roughly as: "No propositions can be produced of which it can be shown that they can neither be proved constructively nor disproved constructively". For the non-constructivist this would seem to leave open the possibility that there are absolutely unsolvable problems A "out there", but we cannot produce ones of which we can show that they are unsolvable. (2006, p. 147)

Feferman's point here is that while Martin-Löf's argument succeeds at establishing optimism for the constructivist, it falls short of establishing optimism *tout court*. He goes on to present examples of problems that are "absolutely unsolvable from the standpoint of practice" (Feferman, 2006, p. 149).

Feferman argues that the non-constructive mathematician can evade Martin-Löf's target conclusion of optimism by reinterpreting it in a way that fits within a non-constructive worldview. If pessimism or optimism is to be established *tout court*, the reasoning would go, it must be done so independent of a constructive philosophy of mathematics. This can be interpreted in two ways, however. The first emphasizes the constructivist portion of Martin-Löf's reasoning. The second emphasizes the philosophical, where this is understood as something non-mathematically neutral, content of Martin-Löf's argument. For the remainder of this section, we discuss the first interpretation. The second interpretation is the focus of the following section.

The first interpretation emphasizes that Martin-Löf employs constructive understandings of key notions, and that these admit of non-constructive interpretations. We point out only that just because an unorthodox thesis can be given an interpretation that coheres with the orthodoxy does not mean that it should. Even the strictest Quinian should admit that in some cases genuine development first appears as unorthodox. Moreover, it is clear that in some situations translation from the unorthodox is responsible for the loss of relevant content. This is arguably what happens in the non-constructive interpretation of constructivism. In interpreting intuitionistic logic in **S4**, we substitute a constructive understanding of truth for the notion of a proof of something that holds classically. While doing so provides a way of explaining intuitionism to the classical modal logician, it does so at the expense of what is arguably the most foundational notion within intuitionism. In this case also, the practice of recasting constructive contributions as mere features within a broader classical panorama threatens to make unavailable what might otherwise be seen as a genuine scientific advance in Martin-Löf's constructive optimism.

The Axiomatic Method

Based on our discussion of Martin-Löf's argument for constructive optimism and Feferman's response, the question arises: can optimism or pessimism be established on purely mathematical grounds? That is, can we decide this question in some manner that is not seized upon by philosophy?

But what would it be to have such an understanding of mathematics? Perhaps examining a distinction on uses of axiomatics in Gödel can help us to get clear on whether or not it is plausible that there is a way to thus separate off the philosophy. Also in his 1951 lecture, Gödel distinguishes between the proper and merely hypothetico-duductive uses of axiomatics. He claims:

> [The inexhaustibility of mathematics] is encountered in its simplest form when the axiomatic method is applied, not to some hypothetico-deductive system such as geometry (where the mathematician can assert only the conditional truth of the theorems), but to mathematics proper, that is, to the body of those mathematical propositions which hold in an absolute sense, without any further hypothesis [...].
>
> Of course, the task of axiomatizing mathematics proper differs from the usual conception of axiomatics insofar as the axioms are not arbitrary, but must be correct mathematical propositions, and moreover, evident without proof. (1995, p. 305)

As Gödel emphasizes to his audience, of course there are "widely divergent" ways of saying just what counts as mathematics proper. One suggestion might be to use this distinction to try to find a sense of mathematics not seized upon by philosophy.

What happens when we consider proper mathematics axiomatically? To do so is to limit the application of the axiomatic method to a specific domain, setting aside the specific sort of Platonist view that any consistent system has application, as in (Balaguer, 1998). We see this, for example, in the emphasis on contentual reasoning in Sergei Artemov's *Provability of Consistency* (2019), where considerations of meaning filter out gerrymandered uses of formalism. This, of course, is motivated by a philosophy of mathematics and clearly is not philosophically neutral in a general sense.

Let us instead consider the hypothetico-deductive use of axiomatics. By this, after all, Gödel meant reasoning conditionally with axioms irrespective of how they relate to mathematical reality, however that is explained. If the philosophically neutral way of understanding mathematics corresponds to Gödel's hypothetico-deductive use of axiomatics, then this suggests an account of absolute provability as provable in a given hypothetico-deductive system.

There are two objections to this proposal. The first is that it is easy to see that absolute provability as understood in this way would trivialize the notion. In his *Inexhaustibility: A Non-Exhaustive Treatment* (2004), Torkel Franzén makes exactly this point:

> That a formalization of a mathematical statement is provable in a formal theory does not itself imply that the statement can be p r o v e d in the ordinary mathematical sense, that is, that an argument establishing the statement as a mathematical theorem can be given. As an extreme instance, any statement is provable in a theory in which it is taken as an axiom, but this tells us nothing about whether or not the statement can be proved in the ordinary sense. (p. 8)

Anything that can be taken as an axiom is provable in some hypothetico-deductive axiomatic system. Hence if we take absolute provability to be provability in some hypothetico-deductive system, then this trivializes the notion of absolute provability. Second, such a thesis on absolute provability would not give us an account of what counts as mathematical in a way that is not encroached upon by philosophy. To think that what is expressible in hypothetico-deductive formal systems just is this mathematical core itself is of course not philosophically neutral by any means.

There are ways of amending the proposal that absolute provability just is hypothetic-deductive provability. Perhaps we want to consider provability to the standards of a mathematical community (Franzén, 2004), demand that the axioms be objects of knowledge or possible objects of knowledge given certain stipulations (Williamson, 2016, pp. 247–248), that axioms be "deemed plausible" (Clarke-Doane, 2013, p. 469). In any such case though it is clear that insofar as we are appeal to a ground for a given circumscription of proper mathematics that we appeal to philosophical considerations.

While we do not claim that using Gödel's discussion of the axiomatic method is the only way of attempting to find a notion of mathematics not encroached upon by philosophy, the prospects for something else in this vein look dim. The criticism that Martin-Löf's argument is objectionable insofar as it makes use of philosophical notions can seemingly be leveled thus against any account in the literature. For this reason, it seems like it would be misguided. A paragraph from Saul Kripke's *Is there a Problem about Substitutional Quantification?* (1976) makes such a point, though in a different context. He writes:

> Logical investigations can obviously be a useful tool for philosophy. They must, however, be informed by a sensitivity to the philosophical significance of the formalism and by a generous admixture of common sense, as well as a thorough understanding both of the basic concepts and of the technical details of the formal material used. It should not be supposed that the formalism can grind out philosophical results in a manner beyond the capacity of ordinary philosophical reasoning. There is no mathematical substitute for philosophy. (1976, p. 416)

An answer to the question of pessimism or optimism does not seem to be the sort of thing that can be achieved by recourse to mathematics that is not seized upon in some sense by philosophy. Instead, the answer to this question is a consequence of an account of absolute provability, which is itself a thoroughly philosophical undertaking. Martin-Löf's reasoning cannot be faulted for essentially attempting to do just this.

The Third Law

We suggested that two sorts of responses to Martin-Löf's argument were unsatisfying. The first was that his argument relied upon constructive notions, which admit of a non-constructive interpretation. This, we argued, is to overlook the revolutionary character of constructivism, and in this way was less of an objection than a dismissal. The second was that Martin-Löf's argument relied upon non-mathematical groundwork. We argued, however, that we should be skeptical about the possibility of finding an account of mathematics that is not somehow permeated by philosophy. Setting the above discussion aside, when we introduced Martin-Löf's argument we did flag a premise for later discussion.

This was Martin-Löf's third law, that from the fact that it was not possible to know a given proposition, we can conclude positive evidence for the negation of that proposition.

How is this justified? Take a given proposition *A*. By *a:A*, we mean that *a* is a proof of *A*. By the claim that *A* cannot be known, Martin-Löf understands that "the situation *a:A* cannot arise, for any *a*" (Martin-Löf, 2013, pp. 11, 13). He continues:

> Now, from this negative piece of information, I have to get something positive, namely, I have to construct a refutation, and a refutation of *A* is a hypothetical proof of [*falsum*] from *A*, or, equivalently, a function which takes a proof of *A* into a proof of [*falsum*]. The argument is this: we simply introduce a hypothetical proof of [*falsum*] from *A*, call it *R*. (2013, p. 13)

The thought is that we have negative information that it is impossible that for any *a*, it holds that *a:A*. We get the positive refutation of *A* by constructing a hypothetical proof of *falsum* from *A*.

But what does it mean to say that it is impossible to know that *A*, or alternatively, that the situation *a:A* cannot occur for any *a*? Martin-Löf is clear about what he means by possibility. He writes: "[By] the notion of possibility, I have nothing more to say, except that it is the notion of logical possibility, or possibility in principle, as opposed to real or practical possibility, which takes resources and so on into account" (2013, p. 9). Since Martin-Löf is here discussing applications of introduction and elimination rules, it seems clear that his "possibility in principle" or "logical possibility" will have to do with what can be reached by transformations of this sort.

A first objection is that this view assumes that the rules articulated by some specific system express what it really is for something to be possible in principle. After all, if our set of rules is somehow suspect, it would be strange to assume they were even in the position to lead us securely to a mathematical insight. But perhaps we can make this point sharper. In his *Science and Method* (2012), Henri Poincaré discusses the dynamic nature of the concept of solution:

> Many times already men have thought they had solved all the problems, or at least that they had made an inventory of all that admit of solution. And then the meaning of the word solution has been extended; the insoluble problems have become the most interesting of all, and other problems hitherto undreamt of have presented themselves. (2012, p. 370)

The thought here is that the horizon of what seems possible at a given period is consistently surpassed, and that it is in these instances that we see genuinely interesting development. After this, we revise what we thought was possible.

There is perhaps a stronger objection, though, to Martin-Löf's third law. We can concede that the specific laws chosen of the system in question actually do characterize logical possibility in the relevant sense. Nonetheless, the third law involves the passage from a logical insight to a mathematical one insofar as we go from a fact about logical impossibility to a positive refutation of a claim. There is something non-constructive about this. To get clear on this, we look again to the thought of Poincaré. Michael Detlefsen dubs "Poincaré's Principle of Epistemic Conservation" the thesis that "there can be no increase in genuine knowledge of a specific mathematical subject without an underlying increase in subject-specific insight into (i.e. intuitional grasp of) that subject" (Detlefsen, 1990, pp. 501–502). While logical reasoning is characteristically general, mathematical understanding arguably involves subject-specific insight. But Martin-Löf's third law, insofar as it relies on a notion of logical impossibility, takes us from a general claim about what can be done, in this case with the application of introduction and elimination rules, to the existence of a positive mathematical insight. While perhaps in some cases the realization that a proof of some proposition is logically impossible will lead to a specific proof of the refutation of that claim (consider the case in which the proposition in question is one about the capabilities of the rules that characterize this notion of possibility), to assume that this holds generally is far stronger. Indeed, that there could be a general recipe for getting mathematical insights from logical ones is exactly the sort of thing that contradicts Poincaré's Principle.

Perhaps we can fix a notion of possibility in a different way. For example, we might consider what an actual agent can do or what an idealized agent can do. For the remainder of this section, we argue that in both cases we need not endorse the third law. The first sort of case suggests considering empirical agents who fall short based on lack of resources or similar circumstances. While this is not the sort of agent Martin-Löf has in mind when discussing possibility, it is worth examining nonetheless. In terms of possible worlds, this is discussed in the context of the condition that if A holds in all worlds accessible from a given world in a Kripke model, then at that world A is known, in Sergei Artemov's *Knowing the Model* (2016, p. 4). Let A be some arithmetical truth unknown to a particular individual. Assume that they have seen no proofs of A. It would seem strange for that individual to conclude the negation of A from the agent's limited information. Alternatively, consider a B that is refutable. The agent, of course, has seen no proof of B. Even if they correctly conclude the negation of B, their reasoning is too hasty; something is missed when they move from a claim about their own abilities to an actual refutation of B. It thus seems that if one wants to understand the relevant notion of possibility invoked in the third law as "possibility for an empirical agent", they need not endorse the third law.

The above has to do with the sorts of mistakes empirical agents are prone to make. What if we were to idealize away from these concerns? Even if we consider the subject divorced from such empirical limitations, it would not follow that the agent would be fully aware of their own capabilities in the sense required for a version of the third law. For example, in his *On the Fourfold Root of the Principle of Sufficient Reason* (1997), Arthur Schopenhauer argues:

> [T]he subject knows itself only as a willer, not as a knower. For the ego that represents thus the subject of knowing, can itself never become representation or object, since, as the necessary correlative of all representations, it is their condition. (1997, p. 208)

Schopenhauer's subject is clearly not a limited empirical subject. Nonetheless, the subject inasmuch as it is a subject cannot know itself, because to know itself would be to treat itself as an object qua object of knowledge. The subject is explicitly ignorant of itself as a knower and must be so for the above reason. Again, it would seem strange from the subject to conclude from the impossibility of knowing that it is a knower to the conclusion that it is not a knower. Here we see then that even when a subject is considered in a way that has been idealized away from empirical limitations, the third law need not be accepted.

Conclusions

In this paper we discussed Martin-Löf's argument for constructive optimism. Therein, he argues for a form of optimism based a view of absolute provability as knowability. We presented criticisms of some objections to Martin-Löf's argument. Then, we put forth a novel criticism of Martin-Löf's argument based on his third law. His third law is a general rule describing the passage from a point about the impossibility of a proof, where this is understood in terms of logical possibility, to the positive existence of a refutation. This, we argued, ran afoul of Poincaré's Principle. A possible emendation of the third law was to interpret the notion of possibility therein in a way that does not appeal to logical possibility. An intuitive thought, especially in the constructivist context, is to think of possibility in terms of the abilities of agents. But even in this case, we continued, the third law need not hold. While the reasoning against the third law was straightforward when the agent under consideration has the limitations of an empirical agent, the Schopenhauerian subject provided an example of a case in which there are important reasons that even an idealized subject's abilities might not be completely transparent to them. In both instances, we observe failure of the third law as a general principle of reasoning.

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