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Article

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THE NOTION OF EXPLANATION IN GÖDEL'S PHILOSOPHY OF MATHEMATICS[1]

Summary: The article deals with the question of in which sense the notion of explanation (which is rather characteristic of empirical sciences) can be applied to Kurt Gödel's philosophy of mathematics. Gödel, as a mathematical realist, claims that in mathematics we are dealing with facts that have an objective character (in particular, they are independent of our activities). One of these facts is the solvability of all well-formulated mathematical problems—and this fact requires a clarification. The assumptions on which Gödel's position is based are: (1) metaphysical realism: there is a mathematical universe, it is objective and independent of us; (2) epistemological optimism: we are equipped with sufficient cognitive power to gain insight into the universe. Gödel's concept of a solution to a mathematical problem is much broader than of a mathematical proof—it is rather about finding reliable axioms that lead to a (formal) solution of the problem. I analyse the problem presented in the article, taking as an example the continuum hypothesis.

Keywords: mathematical realism, mathematical explanation, incompleteness theorems, mathematical universe, continuum hypothesis.

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One of the theses put forward by Gödel is that about the solvability of all well-formulated mathematical problems. From the point of view of experience in the field of "everyday" mathematics (including school mathematics), this thesis seems obvious: every task can be solved, even very difficult open problems eventually give way to the pressure of the efforts of generations of mathematicians. However, Gödel is the author of the theorem that for each (reasonable) theory T, there are propositions that are undecidable in this theory. How can we reconcile this result with his thesis on the universal solvability of problems? In order to answer this question, a certain explication of the concept of solving a mathematical problem is necessary. Then it will be possible to analyse the thesis, according to which every problem would be solvable. How to explain it—and what explanation for this state of affairs is given by Gödel? I think that using the category of explanation here is justifiable. It is more and more often discussed in relation to mathematics—here it will have some specificity, but I think that its use will shed new light on the issue.

The article has the following structure:

1. Gödel's philosophy of mathematics.
2. The problem of explanation in mathematics.
3. The example of the continuum hypothesis.
4. Summary.

In part 1, I point to the basic elements of Gödel's philosophical worldview. The presentation is of course—necessarily—brief. In Part 2, I formulate the basic questions posed in the debate, I also briefly mention the problem of mathematical explanations in the natural sciences—and I formulate the title question/s. Part 3 is devoted to the analysis of the issue on the basis of a standard and well-known example—namely the continuum hypothesis. The article ends with a short summary.

1. Gödel's Philosophy of Mathematics[2]

Gödel was in a way, a model mathematical Platonist. In his opinion, there is an objective, mathematical universe independent of us, which is

[2] This is a very brief and sketchy presentation. A detailed analysis of Gödel's philosophical position is contained, for example, in the works of Krajewski (2003) and Wójtowicz (2002).

described (although, of course, in an imperfect way) through mathematical theories—and to which we have cognitive access through a kind of intuition.[3] Gödel focused on set theory, and his philosophical analyses often refer to it.[4] Gödel's views on the nature of mathematics naturally combine with a broader vision regarding the role and nature of philosophy. Gödel stressed the importance of fundamental analyses, in particular, analyses of the meaning of basic metaphysical concepts. He even hoped that he could describe these terms in an axiomatized way.[5] It is worth emphasizing his clear opposition to the dominant neo-positivist vision of mathematics (and philosophy, in particular metaphysics). Gödel even argued that the "spirit of the times" (*Zeitgeist*) is not in favour of his views that metaphysical considerations are meaningful and that mathematics is not the syntax of the language of science, but expresses objective truths. Conventionalism is not a good explanation for the nature of mathematics; conventions are, of course, present in mathematics, but they are not arbitrary, but—freely speaking—they convey the essence of concepts and express objective truths.[6]

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[3] "Despite their remoteness from sense experience, we do have something like a perception also of the objects of set theory, as is seen from the fact that the axioms force themselves upon us as being true. I don't see any reason why we should have less confidence in this kind of perception, i.e., in mathematical intuition, than in sense perception, which induces us to build up physical theories and to expect that future sense perceptions will agree with them, and, moreover, to believe that a question not decidable now has meaning and may be decided in the future" (Gödel, 1964, pp. 120–121).

[4] Gödel's philosophical worldview was clearly reflected in his methodological decisions regarding how (by which methods) mathematics can be practised. Gödel declared that the belief in the existence of an objective mathematical world constituted the motivation for the free use of non-constructive methods based on strong assumptions about the existence of objects of a certain type.

[5] Gödel's proofs for the existence of God can be considered an attempt at this type of precision. Wang talks about the conversation between Gödel and Carnap on the 13th of September, 1940 (1987, p. 217), the subject of which was metaphysics, in particular the creation of a coherent metaphysical doctrine based on the notions of God and the soul as primitive. In Carnap's opinion, such a theory would have a mythological character, whereas Gödel's position is completely different. He claims that such a theory could be no less sensible than theoretical physics, which cannot be expressed in purely observational terms.

[6] The discussion of "syntactic interpretation" is devoted, for example, to the work in which Gödel writes: "in whatever manner the syntactic rules are formu-

Sometimes Gödel's position is presented as an expression of some kind of dogmatism—through a certain type of "act of faith" we postulate the existence of a mathematical universe to which mathematical propositions refer. Such a position would resemble the "working hypothesis" of many mathematicians—those who to the eternal question of whether mathematics is discovered or created, answer discovered (which is consistent with the position of realism, and can even be interpreted as one formulation of the realistic position). It would be an expression of a certain type of natural ontological position of a mathematician—but without any further justification.[7] However, Gödel did not accept this position in a dogmatic or non-reflective manner. It is worth noting a quite unusual—considering the conception of Gödel—and probably little-known quotation: "Our axioms, if interpreted as meaningful statements, necessarily presuppose a kind of Platonism, which cannot satisfy any critical mind" (Gödel, 1933, p. 50). We do not find such sceptical statements very much, but they document the fact that Gödel was aware that accepting a realistic position requires justification (and, of course, more precision—because realism can take many different forms). This may testify to a certain evolution of Gödel's views. He writes very clearly about this:

> Some body of unconditional mathematical truth must be ackhnowledged, because, even if mathematics is interpreted to be a hypothetical-deductive system, still the propositions which state that the axioms imply the theorems must be unconditionally true. The field of unconditional mathemati-

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lated, the power and usefulness of the mathematics resulting is proportional to the power of mathematical intuition necessary for their proof of admissibility. [...] it is clear that mathematical intuition cannot be replaced by conventions, but only by conventions plus mathematical intuition" (Gödel, 1953/9, p. 358).

[7] "However, when I do mathematics, I have a subjective feeling that there is a real world to discover: the world of mathematics. This world is much more imperishable for me, immutable and real than the facts of physical reality" (L. Bers, in: [Hammond, 1978, p. 19]). Hardy: "Personally, I always considered the mathematician in the first place as an observer, a man who observes a distant mountain range and notes his observations. His task is to clearly identify and describe to others as many peaks as possible" (Hardy, 1929, p. 18). Cantor talked about himself as a rapporteur for the results of his research. The conviction that the world of mathematical entities exists objectively—and we only discover it—connects all these mathematicians. Of course, I'm not saying that this position is the only one—or even that it is the dominant position among mathematicians, but that is a separate issue.

> cal truths is delimited very differently by different mathematicians. At least eight standpoints can be distinguished. [...]: (1) classical mathematics in the broad sense (i.e. set theory included), (2) classical mathematics in a strict sense, (3) semi-intuitionism, (4) intuitionism, (5) constructivism, (6) finitism, (7) restricted finitism, (8) implicationism. (Gödel, 1953/9, p. 346)

Gödel's argumentative strategy consists in adopting a weak version of realism as an initial assumption—and then a gradual strengthening of the position by indicating the relevant arguments. Number theory is a natural choice for this initial assumption because it is fundamental in mathematics—and widely known. Number theoretic propositions seem to express objective content.[8] The assumption that number-theoretic propositions have an objective character seems to be relatively uncontroversial. This is clearly stated in the following quote:

> Logic and mathematics—like physics—are based on axioms that have real content [...]. That such real content exists is evident through the study of number theory. We come across facts that are independent of any conventions. These facts must have content, because the consistency of number theory cannot be based on trivial facts. [...] There is a weak form of Platonism that no one can deny. [...] When we compare the Goldbach hypothesis with the continuum hypothesis, we are more convinced that the first of them must be true or false. (Gödel's statement in: Wang, 1996, pp. 211–212)

This opinion is significant in the context of Gödel's first and second theorems, according to which Peano arithmetic (PA) is imcomplete and its own consistency cannot be proved. Gödel's sentence (constructed in the proof) expresses—freely speaking—its own unprovability. We perceive it as true, but of course this is already due to a semantic analysis, going beyond the formal PA arithmetic. According to Gödel, such argumentation is fully legitimate (although it is not formalizable in PA). The source of mathematical knowledge is the analysis of concepts. It is based on the specific cognitive ability of our mind, i.e. mathematical intuition. This leads us to ever stronger theories, which we have the right to give realistic interpretations.

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[8] It seems relatively natural to recognize that the truths of number theory have a "hard" character, that they are not just a matter of convention. The thesis that there exist n! permutations of the n-element set seems to be objective—and not the result of a purely conventional assumption.

2. Problems of Explanation in Mathematics

The problem of mathematical explanation is found in (at least) two areas: (i) mathematical explanations in natural sciences; (ii) explanations inside mathematics. Here, I focus exclusively on the issue of (ii). Probably the most natural version of this issue is the question about the explanatory nature of mathematical proofs: can (should?) mathematical proof play an explanatory role—and what does that mean? It is clear that the basic function of proofs is to convince (in accordance with the standards of mathematical argumentation) that a theorem is true. At the same time, the natural (but not strictly formal) question for a mathematician is one about deeper causes, about the whole "background of phenomena". Speaking freely, when analysing mathematical proofs, it is important not only how the individual inferential steps follow from each other, but "what's really going on here?". Using somewhat metaphorical language, it is about this subtle "game of mathematical concepts", which does not boil down to the fact that the next step of the proof results from the previous one. Understanding mathematical proof as a formal verification of facts (by examining formal dependencies) does not fully reflect the understanding of mathematical proof as a source of mathematical knowledge. Sometimes mathematicians speak in such a spirit:

> Even when a proof has been mastered, there may be a feeling of dissatisfaction with it, though it may be strictly logical and convincing; [...]. The reader may feel that something is missing. The argument may have been presented in such a way as to throw no light on the why and wherefore of the procedure or on the origin of the proof or why it succeeds. (Mordell, 1959, p. 11; citation based on: Mancosu, 2008, p. 142)

Similarly, Rota writes (in the context of computer evidence) that "[v]erification is proof, but verification may not give the reason" (Rota, 1997, p. 187).[9] The question about the explanatory role of mathematical proofs has a long history—as early as in Aristotle one can find a distinction corresponding, in today's terminology, to reasoning that only justifies

[9] There is no room for detailed analysis of the issue. I consider Rav's article (1999), in which the author analyses the role of proofs in mathematics, accentuating its central place, to be very interesting.

a certain thesis and reasoning that explains the reasons.[10] Mathematicians themselves are obviously aware of the different nature and function of proofs. Mancosu (2018) gives an example of a monograph on algebraic geometry, which deals with various proof methods, and in which the author rejects the so-called the transfer method (despite its effectiveness), indicating that it allows to give a logical proof of a certain result, but does not explain it.[11] The discussion about explanations in mathematics is lively—there are many detailed analyses regarding individual theorems, the links between the problem of explanation and the (quite elusive but important) concept of depth in mathematics,[12] aesthetic issues, or the problem of purity of proofs (i.e. using methods limited to a given field—e.g. purely geometric methods in proofs of geometry theorems or combinatorial methods in combinatorial proofs). However, there is still no good general answer to the question of what is the real source of explanatory power of mathematical proofs.

The problem of explanation may also have a broader character—and may relate not only to the proofs, but even to broader classes of issues. The question "why is squaring the circle impossible?" has a slightly broader dimension: the answer can be found outside of geometry, in Galois's theory. Therefore, it is no longer a question of the proof only, but also or giving a proper interpretation of one theory in another. Similarly, you can ask questions about the nature of concepts which are fundamental for a given theory, about the most natural formulations (definitions), etc. This is a very broad issue and will not be addressed here.

This problem of explanation (or maybe: a series of problems) concerns explanations inside mathematics. However, the subject of analysis in this article is a question that is not mathematical *par excellence*—rather philosophical or methodological. The general question about why every mathematical problem is solvable has a completely different character to the very specific question, for example, why every differentiable function is continuous, or why squaring a circle or triscecting an angle is not possible.

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[10] See, for example, Mancosu (2018), where the reader will find a detailed description of the problem of mathematical explanations (both in physics and in mathematics itself) together with a comprehensive and up-to-date bibliography. I thank one of the reviewers for drawing my attention to this.

[11] This monograph is Brumfiel (1979). In another work (Hafner & Mancosu, 2008), the authors analyse this example in the context of Kitcher's explanation theory.

[12] See special issue *23*(2) *Philosophia Mathematica* (2015).

In these cases, we primarily ask about the proof, or possibly its analysis and commentary (explanation): what "resources" we use, what assumptions are necessary (and what role do they play in the proof), which set of concepts we refer to, what is the "conceptual environment"? Ultimately, therefore, often the answer can be reduced to analysing some specific proof. On the other hand, it is difficult to expect a similar analysis of a philosophical thesis—especially in the context of the fact that the Gödel's first theorem seems to contradict this thesis at first glance.

However, in the context of Gödel's philosophy of mathematics, I consider using the concept of explanation in this context to be legitimate. The concept of solving a mathematical problem—according to Gödel exceeds the notion of formal proof. It should be remembered that Gödel considered technical and philosophical issues to be intimately connected.[13] It is worth recalling that Gödel believed that philosophical considerations could be given a clear form and that (after sufficiently good clarification of the concepts) philosophical discussion reaches the level of precision which is typical for mathematics (Gödel, 1951, p. 322). In such an optimistic spirit, one can interpret his statement that the design of Leibniz's *characteristica universalis* was not a pure utopia (Gödel, 1944, p. 101). At the same time, he admitted that this is a matter for the future and that, for the time being, philosophy has not reached a sufficient degree of development (Gödel, 1951, p. 311). He himself admitted that he did not give his analyses a sufficiently precise form.

We talk about explanation in a natural way when we are dealing with a phenomenon that we want to describe, understand or just explain. Usually (and certainly often) this phenomenon is something external, it is not a convention, for example physical phenomena are given to us, we are confronted with them. Will a similar approach be appropriate for mathematics, which seems to be our creation, though? In the context of Gödel's realistic position, such an approach is natural: mathematics is somewhat independent of us, it has an objective character. So it is not surprising that we are confronted with objective facts—also concerning mathematics. We want to explain these facts. An example of such a fact is the solvability of problems. Answering the question: "Why is every well-formulated mathematical problem solvable?" is associated with the need to clarify

[13] The creator of set theory, Cantor, argued that mathematical and philosophical problems cannot be separated—and that set theory would give a theological interpretation (e.g., Murawski, 1984; Purkert, 1989).

how to understand the concept of solvability (solution) of a mathematical problem. This issue can be "invalidated" by reducing it to a king of tautological statement: the problem is well-formulated exactly when it is solvable (even if we do not know this solution, or even—potentially—we will never know it). And here the discussion ends. However, I believe that would not be the right attitude to the matter. The concepts of "well-formulated problem" and "solution to the problem" are not easily reducible to each other—the history of mathematics shows clearly that it would be an over-simplification.

The concept of solving a mathematical problem from the point of view of ordinary, everyday mathematics has obvious meaning: to "solve the problem" is simply to provide the appropriate proof, using standard means. Probably for 99.9% of problems encountered by a mathematician in practice, this is what is meant by a solution. However, the situation becomes more complicated when we reach problems which are undecidable within standard mathematics. The question arises what standard mathematics is. The view that standard mathematics can be reconstructed in ZFC set theory (i.e. Zermelo-Fraenkel set theory with the axiom of choice)—and it is the ZFC that sets the framework of the "mathematical standard"—is quite common in the philosophy and foundations of mathematics. This point of view is very clearly visible in Gödel himself.

It has been known from the moment of proving Gödel's first theorem that ZFC is an incomplete theory, and the first example of an independent proposition with a clear mathematical content is the continuum hypothesis.[14] It is obvious, therefore, that the concept of solving a mathematical problem must have a different meaning to "deciding it within ZFC"—otherwise Gödel's thesis would be clearly and obviously false.

Gödel's position is worth considering in the context of Hilbert's programme and Hilbert's mathematical worldview. Hilbert was undoubtedly a cognitive optimist—he argued that there is no *ignorabimus* in mathematics and that any well-formulated mathematical problem can be

[14] Gödel's theorems talk about the existence of independent propositions, but the construction of Gödel's sentence does not lead to propositions with a natural mathematical content. CH is such a natural sentence which is independent of ZFC—and this is a very important result. It is worth adding that the first independent propositions from PA with a clear combinatorial content were given only in the 1970s (Paris & Harrington, 1977).

solved.[15] Hilbert's programme can also be seen as an expression of this optimism: he hoped to find a safe foundation for mathematics—which would also be strong enough to solve (all) well-formulated problems. Tools for this are to be provided by proof theory . Hilbert was, therefore, convinced that any mathematical problem could be solved in a literal sense (probably closest to the colloquial meaning).[16]

A common assertion in the literature is that Gödel's theorems dealt a fatal blow to Hilbert's programme. This is a suggestive statement, but probably Gödel would not agree with it himself, in any case not entirely. In his unpublished notes, he notes that interpreting finitist mathematics as a purely formal system leads to a dilemma (Gödel, 193?, p. 164). We can, therefore, say:

(i) that not every mathematical problem is solvable;

(ii) that the syntactic approach to proof does not constitute a proper representation of our concept of proof as something that is the source of our certainty and allows the solving of mathematical problems.

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[15] The French physiologist, Emil du Bois-Reymond, in 1872, formulated the thesis of *ignorabimus*, according to which science is burdened with internal limitations, and so there must be problems impossible to solve. His brother was Paul du Bois-Reymond (an eminent mathematician) who considered this thesis also justified in relation to mathematics (McCarty, 2004). This brings Kant's attention to the questions agonising people's minds, which "one cannot suppress, because he is asked it by his own nature, but which he cannot answer because they outweigh all his potency" (Kant, 1957, p. 7).

[16] Slightly simplifying, it can be said that up to the turn of the 19th and 20th centuries there was no concept of formal proof, and mathematical proofs had—speaking freely—a semantic character. Only with the development of formal logic was it possible to formulate the concept of "formal proof" as a specific set of operations with a formal character (although beliefs of this type—in a yet undefined form—were already present in mathematics). A paradigmatic example, which very clearly shows the discrepancy between the traditional (semantic) and formal concept of proof, is geometry, which was formalized by Hilbert in *Grundlagen der Geometrie*. The formalistic point of view on geometric proofs obviously assumes that there is some established formal system in which these proofs are reconstructed and that this system encompasses all truths (or "truths"). There is no room for intuitive argumentation—for example Hahn was very radical against the concept of intuition.

Gödel points to the fact that

> number-theoretic questions which are undecidable in a given formalism are always decidable by evident inferences not expressible in the given formalism. As to the evidence of these new inferences, they turn out to be exactly as evident as those of the given formalism. So the result is rather that it is not possible to formalise mathematical evidence even in the domain of number theory, but the conviction about which Hilbert speaks remains entirely untouched (Gödel, 193?, p. 164)

thus advocating the second possibility. It can be said that, in his opinion, the syntactic interpretation leads to the loss of important aspects of the proof.

And just seeing this fact allows Gödel to remain a cognitive optimist with regard to mathematics. However, he interpreted the concept of "solution to a mathematical problem" in a radically different way from Hilbert. According to Gödel, convincing mathematical reasoning can be informal.[17] An example is the proposition constructed in the proof of Gödel's theorem: there is no doubt that the proposition "I am unprovable within PA" is perceived as true, although of course it is not provable within PA.

So, the notion of "resolving a mathematical problem" will be interpreted by Gödel in a very different way from Hilbert. It can be said that they interpret the term "mathematical knowledge" in a different way, or that they respond in a different way to the question "what does it mean to have mathematical knowledge?" From the point of view of the Hilbert programme, obtaining mathematical knowledge is possible thanks to the establishment of an unquestionable, finitary fragment of mathematics (and then by performing the appropriate theoretical reduction). For Gödel, the matter looks completely different—which is of course related to the incompleteness theorems. No formal theory (satisfying the relevant natural conditions) is a complete theory, and thus it will not be possible to solve all mathematical problems in one theory.) The process of obtain-

[17] It is worth mentioning again that, according to Gödel, it will be possible to conduct a philosophical discussion with mathematical accuracy (the condition is a good explanation of concepts; Gödel, 1951, p. 322). Wang cites Gödel's opinion that a precise metaphysical doctrine will be formulated in the future. Its absence results from the erroneous way of practising philosophy (and theology) as well as the prevailing scientistic superstitions (Wang, 1987, p. 159).

ing mathematical knowledge goes beyond formal procedures, and mathematical argumentation is not reducible to the concept of "proof in theory *T*". The proofs that we know from mathematical practice, of course, are not formal in nature: rather, they consist of convincing arguments in which an intuitive understanding of mathematical concepts is inevitably present—not only formal transformations. A spectacular example is the proof of Fermat's theorem—it is hard to imagine what it would look like in a fully formalized version, but it certainly would not be readable for us.[18]

The central concept in Gödel's philosophy of mathematics is mathematical intuition—a kind of intellectual ability to recognize mathematical truths, that goes beyond the mechanical manipulation of symbols. In this context, it is worth mentioning the important work of Turing (1939). Turing draws attention to the fact (in the context of Gödel's results) that we are able to see the truth of unprovable statements in a given formalism. In his work, he analyses the problem of the whole system of increasingly stronger logics, in which it will be possible to solve ever-wider classes of mathematical problems—which can also be understood as a technical equivalent of Gödel's idea going beyond the given formal system.[19] Regardless of how we are going to understand the concept of mathematical intuition, there is no doubt that it cannot be mechanical—and thus cannot be "imitated" in the standard model of the Turing machine. However, it can be argued (e.g., Hodges, 2013) that the concept of the oracle, introduced by Turing, is the formal equivalent of cognitive activities that go beyond mechanical procedures. Turing does not analyse the nature of the oracle in more detail, limiting himself to the statement that it cannot be a machine. It can, therefore, be said that the informal, intuitive component of the activity of the mathematician has been "incorporated" into the technical definition here.

There is a tension here between what we would call a "mathematically convincing argument" and its formal paraphrase (or perhaps: its *explica-*

[18] An interesting example of a proof that is short, understandable and fully acceptable is given by Boolos (1987). This is a proof in second order logic—but the formalization of this proof in first order logic would be "astronomical" in length. The problem of formalizing this proof in Mizar is the subject of analyses in the work of Benzmüller and Brown (2007). I thank one of the reviewers for drawing my attention to this issue and for the bibliographic suggestions—as well as for suggestions regarding Turing's work.

[19] In Marciszewski's essay (2018) this issue is discussed more comprehensively.

tum in the form of the concept of formal proof). The formalistic position (in the wide sense) reduces the notion of a mathematically correct argument to the notion of a formal proof in the relevant theory T. However, Gödel's position is completely different—from his point of view, well-formulated mathematical problems are not problems that are solvable within some specific theory T. Rather—freely speaking—for each well-formulated mathematical problem one can formulate the relevant theory T that will solve it. And, of course, it is not a trivial claim that if we have a proposition φ independent of the theory T, then within the theory $T + \varphi$ (i.e., T with φ added as a premise), this problem will be settled. The point is, of course, that it is possible to search for natural, mathematically justified theories T^*, being extensions of T—and resolving our (previously) undecidable propositions.

It is worth mentioning the discussion between Gödel and Zermelo regarding, *inter alia*, the issue of solving mathematical problems.[20] In a letter to Gödel of 21st September, 1931, Zermelo opposes the thesis that any mathematical notion can be defined by means of a finite series of symbols—he calls this conviction a "finitist prejudice". He even claims that Gödel's results express an obvious fact: if only countably many sentences can be defined in a formal language, and there are uncountably many truths, then obviously there must be unprovable truths. It can be argued that Zermelo underestimated the importance of Gödel's results and did not fully understand the technical subtleties. Gödel responds to Zermelo's letter (in a letter dated 12.10.1931), explaining what the essence of his proof consists of—and in particular, emphasizing that what is relevant are statements expressible in a given system, but unprovable in this system, and at the same time provable in a more powerful system. Zermelo interprets the use of a stronger system as a modification of the concept of proof itself. He argues that providing proof involves making the proved sentence obvious, which is achieved by formulating a suitable set of propositions. Zermelo poses a question about what this obviousness is—and at the same time formulates the hypothesis that in a suitable system every mathematical problem is solvable (letter to Gödel from 29.10.1931). The correspondence did not go any further, however, it is an interesting testimony to the early reception of Gödel's results. Another

[20] I thank one of the reviewers for drawing my attention to this issue, and for pointing out the work of Ebbinghaus, Fraser, Kanamori (2010), in which (on pages 482–501) the correspondence cited is included.

interesting point is the issue of problem solving: Gödel, being aware of the existence of metamathematical constraints, believes that it will be possible to establish new axioms that allow for the resolution of subsequent problems. On the other hand, according to Zermelo, these limitations are an obvious defect of the finitist systems, and mathematical reasoning should be reproduced in infinitary systems. This is in accordance with his well-known statement that the proper logic for mathematics is infinitary logic.[21]

3. The Example of the Continuum Hypotheses

Gödel distinguished between objective mathematics (as a set of truths about the mathematic universe) and subjective mathematics (i.e., that which is known to us). His realistic position assumed that the task of the mathematician is to search for a description of mathematical reality—which is objective and exists independently of us. Formal systems describe it only partially—and of course we cannot stop at one particular system as the final set of truths. Rather, it is necessary to analyze mathematical concepts (in particular—the concept of a set) so as to be able to justify new axioms—which will allow for the resolution of subsequent open problems. However, in the case of arithmetic itself, informal reasoning convinces us of the truth of, e.g., Gödel's proposition "I have no proof", while mathematical practice and our beliefs about arithmetic lead to the acceptance of Con(PA). But it would be difficult to give that type of natural and obvious intuitive argumentation in the case of propositions independent of set theory.

In search of an explanation of the solvability of any well-defined mathematical problem it is good to refer to a specific example—and in this article it will be the continuum hypothesis (CH), which is a paradigmatic example of a sentence independent of ZFC.[22] ZFC imposes few limitations: there are many propositions of the type "the value of the

[21] A very interesting description of Zermelo's infinitary logic programme can be found in Pogonowski's work (2006).

[22] The continuum hypothesis is that the power of the set of real numbers (i.e. the power of a continuum) is the smallest uncountable cardinal number, i.e. $\aleph_1$. In another formulation: each infinite subset of R is either countable or equinumerous with R. The independence of CH from ZFC was proven by Gödel and Cohen: Gödel showed its consistency with the ZFC axioms, and Cohen in 1963 the consistency of its negation.

continuum is $\aleph\alpha$" that are consistent with ZFC.[23] However, despite formal independence, one can ask whether there are any convincing arguments that would allow to assign a particular value to the continuum—and above all, whether the continuum problem is a well-posed mathematical problem.

In one of his best-known articles, Gödel analyses the continuum hypothesis (Gödel, 1964). He regards it as an objective, well-formulated question about mathematical reality.[24] It is obviously unprovable in ZFC, but this simply results from the weakness of this theory. For objective mathematics—i.e. all unconditionally true propositions—is one thing, and subjective mathematics: all probative propositions in a given formal theory, is another (Gödel, 1951, p. 305). He himself leant towards the thesis of the falsity of CH, pointing to its paradoxical consequences (Gödel, 1964). However, his views on this matter are not widely accepted. Gödel was, therefore, convinced that it would be possible to find axioms which will determine the value of the continuum. As it is known, the axiom of the constructability V = L implies CH (and also the generalized continuum hypothesis). V = L might be viewed as minimalistic (the universe of collections is "narrow"). So Gödel assumed that it would be possible to prove CH from some axiom of a maximalist character, in a sense opposite to V = L (Gödel, 1964, p. 266). In a certain well-defined sense, large cardinals axioms can be considered to be such maximalist axioms—and here Gödel hoped to find a solution. He was aware that strong axioms of this type would be needed, and that Mahlo numbers relatively low in the infinity hierarchy would not be sufficient.[25]

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[23] There is a well-known theorem that shows how "strangely" the power of cardinal numbers can behave. Easton showed that for any F function meeting two conditions: (1) F is a non-decreasing function from the class of regular cardinal numbers in cardinal numbers; (2) for any κ: $\kappa < \mathrm{cf}(F(\kappa))$; a model for set theory can be constructed in which for any regular cardinal number κ, $2^{\kappa} = F(\kappa)$ (Easton, 1970). In particular, the continuum (that is 2^{ω}) can be large.

[24] Arguments in favour of the thesis that the continuum hypothesis is a well-formulated mathematical problem, not just a metamatematical one, are formulated, for example, by Hauser (2002).

[25] Gödel's article (1964) is not the only (or the first) place where he expressed such opinions. In a lecture at Princeton in 1946 Gödel characterized "strong infinity axioms" as an assumption which, in addition to having a specific formal structure, is "is also true" (Gödel, 1946, p. 151). He also expressed a very optimistic conjecture that "some completeness theorem would hold which would say that

It turned out that this this strategy would not bring success in solving the continuum problem: the results, according to which various strong large cardinal axioms are consistent with both the continuum hypothesis and its negation, are known (Levy & Solovay, 1967). Let us add here that Gödel himself tried to formulate another type of axiom that would solve this problem (Gödel, 1970a; 1970b).[26]

However, regardless of the fact that studies concerning large cardinals did not solve the continuum problem, the very idea of seeking new axioms became an inspiration to researchers, and, the Gödel programme is often referred to in this context. Of course—such axioms could not be *ad hoc*, but they would result from analyses regarding our understanding of the concept of the set and our vision of the mathematical universe. The discussion on this subject is lively—however, even a brief review definitely goes beyond the scope of this article.[27]

So when it comes to the *explicatum* defined above ("solvability of a mathematical problem"), one can be tempted to characterize it as finding the appropriate formal theory T—which is an extension of ZFC—based on natural, acceptable axioms, leading to the formal settlement of the problem P within T. There would be two components here:

- Conceptual-analytical phase: the search for appropriate natural, acceptable axioms—and the formulation of the relevant theory T.
- Technical phase: the resolution of P within T (i.e., standard mathematical work—perhaps very difficult).[28]

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every proposition expressible in set theory is decidable from the present axioms plus some true assertion about the largeness of the universe of all sets" (Gödel, 1946, p. 151).

[26] According to commentators, Gödel's reasoning was mistaken (cf. Ellentuck, 1975; Solovay, 1995).

[27] We may mention, for example: Feferman, (1996; 2000), Friedman (2000), Maddy (1988a; 1988b; 1993; 1997), Steel (2000). Woodin's works (1999, 2001) contain technically very complex methodological analyses, based on which it can be proven that the continuum value is $\aleph_2$. Of course, they are the subject of discussion and controversy, so it cannot be argued that the continuum problem has been solved.

[28] Regarding the continuum hypothesis, he stated: "When the concept of set becomes clear, even when we find satisfactory infinity axioms, there will still be a technical (i.e. mathematical) problem to resolve the continuum hypothesis based on axioms" (Wang, 1996, p. 237).

What is the philosophical background for the belief that this is always possible and that every well-defined problem is solvable? Two important aspects can be identified here. One would be termed metaphysical, and the other methodological. Speaking of the metaphysical aspect, I mean that Gödel's realistic position presupposes the existence of an objective, mathematical universe with a certain nature. Gödel believed that the universe has a set-theoretic character—and that there is one, objective universe in which all mathematical propositions are interpreted, and moreover every proposition is either true or false in it. There are, therefore, no propositions of undetermined logical status, no "shaky" propositions.[29] Gödel's thesis would, therefore, have a metaphysical foundation in a specific vision of the mathematical universe.[30]

Of course, belief in the existence of one objective (though unknown) mathematical universe does not automatically give any clues as to what are the solutions to open mathematical problems. After all, it would be possible to accept the thesis that the mathematical world has an objective and fixed character, but that it is unknowable (that is, the *ignorabimus* thesis would be true, against the optimism of Hilbert or Gödel). And here we touch on the methodological aspect: the way in which we can seek answers to mathematical questions that are *ex definitione* unsolvable within the available, i.e. accepted, standard theory (e.g. ZFC). This is possible by establishing new, credible axioms. Gödel was convinced that our analysis of the concept of set would allow the establishment of such axioms. This is an expression of a specific epistemological vision: accord-

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[29] It would be possible to think this if one adopted the concept of so-called multiverses—i.e. a realistic concept, according to which mathematical reality exists, but it is not a "uniform" mathematical universe, rather the entire "galaxy" of set theoretic universes that implement different concepts of set (e.g. Hamkins, 2012). In such a situation, it would not make sense to say that e.g., the continuum hypothesis has a logical value: in different universes the continuum could take different values.

[30] This article is not of a historical-exegetical nature, but it is worth noting that it seems that Gödel's opinion has undergone some evolution of view. He writes that "it is very plausible that with [V = L one is dealing with an absolutely undecidable proposition, on which set theory bifurcates into two different systems, similar to Euclidean and non-Euclidean geometry" (Gödel, 1939b, p. 155). Thus, he explicitly allows for the existence of absolutely insoluble problems; similar theses can be found in another text (Gödel, 193?). Undoubtedly, he later claimed that V = L should be rejected.

ing to Gödel, we have the ability to analyse concepts and see these truths. He regarded the phenomenological method as promising, and wrote about it explicitly in one of his works (Gödel, 1961; cf. also e.g. Tieszen, 1998).[31]

4. Conclusions

Gödel understands the concept of the solution of mathematical problems much more broadly than as the providing of mathematical proof. Formulating such a proof is obviously a necessary condition (and in the case of the vast majority of standard mathematical problems—sufficient), but there are also mathematical problems for which the formulation of a proof is only the second stage. The first is to find reliable (true!) assumptions on the basis of which this proof can be carried out. Obviously, these assumptions must go beyond the standard set theory, i.e. ZFC.

What though is the explanation for this phenomenon of problem solving? The first assumption on which Gödel's view is based is metaphysical realism: there is a mathematical universe, it is objective, independent of us—and each mathematical proposition has a logical value. The second assumption is a kind of epistemological optimism: we are equipped with sufficiently good cognitive means to gain insight into this universe.

The use of the notion of explanation, which is characteristic of empirical sciences, is justified: in the objectivistic vision of Gödel, we are dealing with facts that are independent of us. One of these facts is the solvability

[31] It is worth mentioning here the "second pillar" of learning mathematical truths—they can be methodological arguments that can be symbolically labelled "fruitfulness". This is a very broad issue that I shall not analyse here. It is worth remembering that Gödel himself very clearly emphasized the importance of this aspect, as evidenced by the following quote: "a probable decision about its [a new axiom—K.W.] truth is possible also in another way, namely, inductively by studying its 'success'. Success here means fruitfulness in consequences, in particular in 'verifiable' consequences, i.e., consequences demonstrable without the new axiom, whose proofs with the help of the new axiom, however, are considerably simpler and easier to discover, and make it possible to contract into one proof many different proofs. [...] There might exist axioms so abundant in their verifiable consequences, shedding so much light upon a whole field, and yielding such powerful methods for solving problems (and even solving them constructively, as far as that is possible) that, no matter whether or not they are intrinsically necessary, they would have to be accepted at least in the same sense as any well established physical theory" (Gödel, 1964, pp. 113–114).

of all well-formulated mathematical problems—and this fact requires explanation.

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