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Article

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GOTTLOB FREGE ON TRUTH DURING THE PERIOD OF THE TWO VOLUME EDITION OF *GRUNDGESETZE DER ARITHMETHIK* (1893–1903)

Summary: In 1893 and 1903, two volumes of the most important of Frege's works *Grundegezte der Arithmetik* were published. This period can be called the peak of Frege's logicism. Although the subject of truth in Frege's logical and philosophical works has been repeatedly investigated, there is a lack of studies on his view in this period, especially in Polish literature.

In this article, therefore, I carry out the following research task: to collect and order Frege's statements about truth during the period of publishing the two volumes of *Grundegezte der Arithmetik*.

I refer to the texts published during Frege's life, published posthumously and his correspondence. Particularly noteworthy are: *Grundegezte der Arithmetik* and the unpublished *Logik* (1897). That is why there are two separate sections dedicated to these two texts, whereas the discussions of truth from other texts are grouped thematically: the problem of antinomy, geometry, expression of generality, and others. The subject of truth appears there in relation to logic, philosophy of language, philosophy of logic, philosophy of mathematics and ontology.

Keywords: Gottlob Frege, truth, logic, truth-values, thought, logicism.

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Introduction

After number, truth is the second great subject in the philosophy of Gottlob Frege. In his over forty years of scientific activity, he repeatedly modified his understanding of it.[1] The changes, however, were not essential, but rather meant seeking the coherence of the proposed understanding of truth and specifying the original intuitions. I would like to add that Frege was always against the definition of truth as the correspondence of ideas [*Vorstellungs*] or a sentence [*Satz*] with reality.

In 1893 and 1903, two volumes of Frege's most important work *Grundegezte der Arithmetik* were published. It was a period[2] that could be called Frege's "peak of logicism", a period in which he believed in the success of his project and worked on its further development.

Although the subject of truth in Frege's logical and philosophical works has been repeatedly investigated, diachronically (Sluga, 2002) and synchronously (Burge, 2005; Dummett, 1981; Greimann, 2003a; 2003b; 2007), there is a lack of studies on his view in the peak period of the development of his logicism, especially in Polish literature, where only scattered comments can be found, focusing on the True as a truth-value, however, the topic is worth a major study.

In this article, therefore, I carry out the following research task: to collect and order Frege's statements about truth during the period of publishing the two volumes of *Grundegezte der Arithmetik*. I refer to the papers published during Frege's life, published posthumously and his correspondence. The subject of truth appears there in the context of logic, philosophy of language, philosophy of logic, philosophy of mathematics and ontology.

Frege's scientific achievements from 1893–1903 are very specific. Between the two volumes of *Grundegezte der Arithmetik* there appeared a small book on numbers in Schubert's approach, rarely mentioned as Frege's book (1899/1990). Of the remaining 29 papers 6 were published during Frege's life, 5 were neither published in his lifetime nor prepared

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[1] For more on this subject see (Sluga, 2002; Besler, 2010, p. 189–201).

[2] Actually, the period lasted until June 16th in 1902, when Frege received the first letter from Russell (Russell, 1902/1976) informing him about the possibility of constructing an antinomy based on Frege's first book (Frege, 1879/1997). Next, Frege himself formulated an antinomy based on the logical system from *Grundgesetze der Arithmetik* (Frege, 1893; 1903). For more on this subject see (Besler, 2016).

by him for publication. The remaining 18 documents are letters, mostly of a scientific nature, written with great care and addressed to eminent scientists of that time: Giuseppe Peano (1858–1932), David Hilbert (1862–1943), Heinrich Liebmann (1847–1939), and Bertrand Russell (1872–1970).

Thus, a total of 32 documents were created by Frege in the period studied here, not all of which, however, refer to the subject of truth. Particularly noteworthy are: *Grundegezte der Arithmetik* (Frege, 1893/2009) and the unpublished *Logik* (Frege, 1897/1983). That is why there are two separate sections dedicated to these two texts, whereas the discussions of truth from other papers are grouped thematically: the problem of antinomy, geometry, expression of generality, and others.

There are a lot of words connected with truth in Frege's writing. Here is a list of the terminology typical for the period studied here, from the first volume of *Grundgesetze der Arithmetik*, along with the number of times each term is mentioned: the True [*Das Wahre*] (150), truth [*Wahrheit*] (112), truth-value [*Wahrheitswert*] (97), the False [*das Falsche*] (78), laws of being true [*Gesetze des Wahrseins*] (12). In addition, Frege often used an adjective (predicate) true/false [*wahr/falsch*].[3] The above terminology also occurs in other texts from the period examined here, with exceptions being indicated.

In the last position from the period examined here (Frege, 1903/2009a), the above words occur less frequently, and the wordings "the False" and "laws of being true" are not present at all. It does not mean that Frege had changed his point of view, but can be attributed to the fact that it is a different type of book. An appendix (Frege, 1903/2009b) was added to the book (Frege, 1903/2009a), and in it an attempt to improve the system after a difficulty formulated by Russell (1902/1976). From the above words appear there (as technical words) only (in one sentence alone): "the True", and "the False".

There are some words fundamentally connected with truth: value-range of a function [*Werthverlauf*], thought [*Gedanke*], contradiction [*Widerspruch*], declarative sentence [*Behauptungsatz*], judgement [*Urtheil*], science [*Wissenschaft*]. It is worth adding that in the paper *Logik* (Frege, 1897/1983) there is no expression "truth-value". Moreover, nowhere in the

[3] Frege's German terminology is translated into English in various ways. Here I rely on the solutions adopted by the editors of the new English translation of *Grundgesetze der Arithmetik* (Frege, 2016). I do not interfere in the translation of quotations. Often—for clarity—I give the original German words.

papers discussed here (or to be more precise, in any of Frege's documents) is there an expression "truth conditions" (*Wahrheitsbedingungen*) often invoked by analytical philosophers referring to Frege's idea of truth (see Dummett, 1981, p. 71; Besler, 2010, p. 76).

In the legacy of Frege, three papers have been found, which are treated as unfinished textbooks on logic: (Frege, 1879–1891/1983), (Frege, 1897/1983), (Frege, 1906/1983). The fourth textbook includes articles published as *Logische Untersuchungen*: (Frege, 1918–1919/1990a; 1918–1919/1990b; 1923/1990). In the last article, Frege presented his point of views on truth, thought, sense and reference, nature of logic, negation, and generality. This subject area corresponds to the subjects of his previous unfinished textbooks on logic (Frege, 1897/1983). In none of the above-mentioned documents is there Frege's logical notation, and their subject matter falls within the scope of philosophy of logic.

It is assumed that *Logik* was written in 1897, between the publication of the two volumes of *Grundgesetze der Arithmetik*. The central theme of *Logik* is truth, as substantially connected with logic.

It seems that dating this paper should not present any difficulties, because Frege gave the date in the sentence: "[...] at noon on 1st January 1897 by central Europe time" (Frege, 1897/1983, p. 135).[4] Moreover, German editors established that Frege mentioned:

1. Wilhelm Wundt's journal *Grundüge der physiologischen Psychologie*, which had appeared since 1874 (p. 144).
2. A review published in 1897 (p. 146).

However, one might be surprised by the similarity of many theses concerning truth and thought to the ones from a much later paper (Frege, 1918–1919/1990a). Here are some possible explanations for this situation, each involving a counterargument:

1. The text was written much later, and the date was not related to the date of writing it. Maybe it was meaningful to Frege for reasons unknown to us. Against this solution is his reference to the review from 1897.

[4] In the whole article the pages numer refer to English translations of Frege's papers. See References for details.

2. Frege did not change his views for twenty years or he returned to previously developed solutions. If so, then the concept of objective thought as something for the question of the True and the False appeared much earlier than the work from the retirement period. Against this solution is the lack of repetition of these theses in other writings from that period, including his letters.
3. Furthermore, there is a lack of the expression "truth-value" in this paper, which was crucial for the examined period.

For the purposes of this article, I assume, however, that *Logik* (Frege, 1897/1983) was written in 1897, between the two volumes of *Grundgesetze der Arithmetik.*

Grundgesetze der Arithmethik (1893)

The task of the first volume of *Grundgesetze der Arithmetik* and the subsequent planned volumes, of which only the second one appeared (Frege, 1903/2009b), was the presentation of arithmetic as developed logic (Frege, 1893/2009, p. VII[5]). Frege wrote there that logic deals with the laws of being true, unlike psychology, which is interested in laws of thought (p. XVI). In this context, the subject of truth appeared from the point of philosophy of language, and—along with the True, the False—as categories used in logic.

The philosophical aspect of truth is presented in *Vorwort*, one of two introductions to the first volume.[6] In the examined period, Frege's philosophy of language was already fully developed and well-established and he referred to his previous article (Frege, 1892/1990b).

He used philosophy of language to characterize truth. The basis was the distinction of (only) three types of linguistic expressions: a proposition [*Satz*], a proper name, and a predicate. Each of these expressions has its sense and the reference (understood as the "object" to which the expression referred).[7] The sense of a proposition is a thought, and its reference

[5] The page numbers referred to the canonical paging of this book, assumed also in (Frege, 2016).

[6] It is an issue for a separate investigation as to why Frege wrote two different introductions, one called *Vorwort*, the other *Einleitung*.

[7] For more on sematic categories in *Grundegezte der Arithmetik* see (Heck, 2010).

is one of two truth values:[8] the True or the False. All true (false) propositions refer to the same object, the True (the False). Here are some examples showing this point of view:

> The names "$2^2 = 4$" and "$3 > 2$" refer to the same truth-value, which I call for short the True. [...] The function $\xi^2 = 4$ can therefore only have two values, namely the True for the argument 2 and -2 and the False for every other argument. (p. 7)

On the basis of the above quotations it can be generally said that every true proposition is a proper name of an object, which is one of two truth-values, being the True in example above. And similarly with false propositions.

Frege arrived at the use of sense and reference in the context of truth from a different side. Content, as an element distinguished from the acknowledgment of the truth, was described by him as judgeable, and he distinguished two more elements (Frege, 1893/2009, p. X) in it:

1. Thought, which is the sense of proposition.
2. Truth-value, which is the reference of proposition.

From an historical point of view the expression truth-value proved to be the most important for his philosophy and logic, in fact for all logic in 20th century. He wrote: "I distinguish two truth-values: the True and the False." (Frege, 1893/2009, p. X)

The truth-value and the number [*Zahl*], but not cardinal number [*Anzahl*], were understood as objective, real, ideal objects. The objects were characterized by the fact that in their own name, meaning a proper name, "[...] they do not [...] carry argument place" (Frege, 1893/2009, p. 7).

It is necessary to add that functions (including propositional functions) do not have a truth-value, because as expressions with a variable they are incomplete. Functions "obtain" their truth-value only when they are completed by arguments. However, then, they are not functions any more. For Frege, a concept is "a function whose value is always a truth-value,

[8] The language of values was introduced into philosophy by Hermann Lotze (1817–1881) and Wilhelm Windelband (1848–1915). Frege was in contact with these academics. Windelband used the expression *Wahrheitswert*, Lotze—*Gedanke*, both of which differed from Frege's understanding of value in logic (Sluga, 2002, pp. 84–85; Besler, 2010, pp. 27–28, 73–81).

the True or the False" (p. 8), for example: the concept of red is actually a function "( ) is red", true for some arguments, false for others.

In a paper devoted to the comparison of his and Peano's notations Frege repeated the above-mentioned notions: "[...] all true sentences [*Sätzen*] mean the same thing, namely the True, and likewise all false sentences mean the same thing, namely the False" (Frege, 1896/1990, p. 240). "I use the word *Satz* in the sense of a combination of symbols whose sense is a thought and whose reference is a truth-value—either the True or the False" (Frege, 1896/1990, p. 242).

Incorrectly constructed propositions are treated as false in Frege's logic and his philosophy of language. (Frege, 1893/2009, p. 10; Frege 1896/1990, p. 230). He gave the following example. He introduced a sign for Sun ʘ and using mathematical language wrote that the sign is greater than 2: "$ʘ > 2$". Frege called such a proposition false because Sun is not a number, however, the following proposition is true: "$(ʘ > 2) \supset (ʘ^2 > 2)$" (Frege, 1896/1990, p. 230). From the definition of the material implication we know that such a formula is true when predecessor and successor are false.

It is necessary to add that not every syntactically correct sentence possesses a truth-value. Frege pointed out two situations:

1. Subordinate clause in indirect speech. Generally, a thought is the sense of a proposition, however, in indirect speech the thought is treated as the reference of the subordinate clause (Frege, 1893/2009, p. X). Thus, the subordinate clause, as a part of indirect speech does not possess a truth-value.
2. Sentence with a proper name without reference like a sentence in poetry (Frege, 1896/1990, p. 227); such a rule was explicitly expressed in a later paper, however, in this period it is also valid (Frege 1897–1898/1983, p. 156).

Frege's views discussed above show that the True is essentially connected with his concept of thought. Actually, not only the True, but the False as well. Frege also wrote about false thoughts, giving the following examples: $0^2 = 4$; $1^2 = 4$; $3^2 = 4$ (1893/2009, p. 6).

According to Frege, the notion of thought, which supplements his categories of the Truth and False, is the meaning of the name of a certain logical value (Frege, 1893/2009, p. 7). Later, he even wrote about the "realm of thought" (Frege, 1918–1919/1990a), considering it an objective reality, unchanging, guaranteeing the possibility of doing science, significantly connected with logic. In the context of logic he wrote: "[...] I express thoughts with my signs, it will be helpful to look at some of the easier cases in the table of more important theorems, to which a translation is appended" (Frege, 1893/2009, p. XI).

Truth is a notion that is also used in Frege's formal logic. For example, logical laws are called laws of being true (Frege, 1893/2009, p. XVI). Some of the logical laws served as the basic laws, not proved in Frege's system, one of them being the problematic law V.[9]

In the period of writing the two volumes of *Grundgesetze der Arithmetik*, the truth-values were used by Frege to determine the conditions for propositions constructed both of connectives and quantifiers to be true. Earlier in this context, Frege used the words: affirm [*bejahen*], deny [*verneinen*] (Frege, 1879/1997, p. 5),[10] instead of the True and the False.[11] There are the logical symbols that Frege characterized with a reference to the truth-value (I give them in Frege's order): judgement-stroke, horizontal-stroke, negation-stroke, equality-sign, quantifier-sign, conditional-sign.

He mentioned his first book (Frege, 1879/1997) and distinguished "two components in that whose external form is a declarative sentence:

1. Acknowledgement of truth.
2. The content, which is acknowledged as true" (Frege, 1893/2009, p. X).

The "acknowledgment of truth" is "marked" on a logical formula by attaching the so-called judgement-stroke and Frege described it as follows:

> We are therefore in need of another special sign in order to be able to assert something as true. To this end, I let the sign "$\vdash$" precede the name of the truth-value, in such a way, e.g., in $\vdash 2^2 = 4$ it is asserted that the square of 2 is 4. (Frege, 1893/2016, p. 9; Greimann, 2000)

[9] For more on this subject see the section *The Problem of Antinomy*.

[10] In the English edition p. 121. Apart from this example, page numbers are given from the English editions.

[11] It is worth adding, that Ernst Schröder (1841–1902), Charles S. Peirce and Frege are treated as originators of truth tables. However, Schröder used the expressions *es gilt*, *es gilt nicht* in this context (Marek, 1993, p. 10–11).

The necessity of this judgment-stroke is so obvious, natural and necessary to Frege that in a letter to Peano, with whom he corresponded during this period, he wrote:

> I have [..] the sign |, the judgement stroke, which serves to assert something as true. You have no corresponding sign, but you acknowledge the difference between the case where a thought is merely expressed without being put forward as true and the case where it is asserted. (Frege, 1896/1976, p. 185–186)

The so-called horizontal-stroke, is a sign of a one-place function from objects whose value is one of two truth-values (Frege, 1893/2009, p. 16–17). The value of this function is the True when its argument is the True. There are two other cases:

1. The False is the function's argument.
2. None of the truth-values is the function's argument, but for example, the number 2 (Frege, 1893/2009, p. 10).

Then the value of the function is the False.

The negation (written as a short stroke attached to the horizontal-stroke) is defined as the value of a false function for every argument and this function without a negation sign is true for every argument (p. 10).

An expression with the equality-sign refers to the True when expressions with the same logical value appear on both sides of the connective and to the False in any other case (p. 11).

The universal quantifier was written by Frege as a concavity in the content-stroke with the Gothic alphabet letter. He assumed that the formula "[...] refer[s] to the True if the value of the function Φ (ξ) is the True for every argument, and otherwise the False" (p. 12).

The conditional-sign (a sign for material implication) was written as a vertical stroke connecting two horizontal strokes and characterized as the False when the predecessor is the True and the successor is not the True (p. 26).

Frege also gives examples of functions whose value for every argument is the False:

1. The formula $\grave{\varepsilon}(\varepsilon = \top\overset{\mathfrak{a}}{\smile}\mathfrak{a} = \mathfrak{a})$ was read as the value-range of a function "it is denied that for every a, $a = a$" (p. 17).

2. Connecting one of the truth-values to a value-range with an equality-sign (see p. 17).

In Frege's logic, truth was also presented by logical entailment. In a paper written between 1899 and 1906 he wrote: "Truths [*Wahrheiten*] can be inferred in accordance with logical laws of inference. If a truth [*Wahrheit*] is given, it can be asked from what other truths its truth follows in accordance with logical laws of inference" (Frege, 1899–1906/1983, p. 168).

To sum up the topic of truth in *Grundgesetze der Arithmetik*, it can be said that it was both a philosophical notion and a useful "tool" for studying the truth-value of logical formulas, inferences and the characteristics of connectives or the quantifier. The True was expressed verbally or using the assertion-stroke.

Logik (1897): An Unfinished Textbook

This paper, unfinished and not published by Frege, is worthy of special attention for a number of reasons, including the method Frege used: referring to the ways of using the word "true" in ordinary language. He also pointed out words associated with the predicate "true", and words that do not have a significant relationship with it, although these expressions were used in an ordinary language. Next, he collected contexts in which the word "true" occurred and rejected misleading, improper usage. He compared the predicate "true" to other predicates (p. 126, 128)[12] and listed differences. The predicate "true" had nothing in common with the ideas [*Vorstellungen*], and was not "applicable to what is material" (p. 126).

Frege suggested setting limits on the valid applicability of the word "true". Although he did not explicitly state this position, it can be said on the basis of this and other papers that the predicate "true" refers to thoughts first and, sentences second, and in particular assertoric sentences [*Behautungssätzen*] (p. 126, 129). For sentences [*Sätzen*] are "a proper means of expression for a thought" (p. 126), and "a sense of the sentence is called a thought" (p. 126).

[12] All quotations from this section, unless otherwise stated, come from the paper mentioned in the heading.

In a natural language, "true" is also combined with ideas and experience, which Frege rejected as groundless. He also wrote that we do not need the word "true" to say that the idea of the Cologne cathedral agrees with reality. As the legitimate use of the word "true" he gave predicating it on a proposition like $2 + 3 = 5$ (p. 129). If, however, one speaks of an idea called true "[...] it is really a thought to which the predicate is ascribed" (p. 126).

Although truth is the goal for all science, logic is in a special way related to the predicate "true", like ethics to "good", aesthetics to "beautiful", physics to "heavy" and "warm", chemistry to "acid" and "alkaline" (p. 128). The word "true" specifies the goal of logic (p. 126).

The "true" and "beautiful" predicates, however, differ significantly. There may be a contradiction between propositions of logic, but "[a]esthetic judgements don't contradict one another" (p. 126). What is true—as Frege wrote—is "true in itself" (p. 126) and what is beautiful is not "beautiful in itself" (p. 126). In addition, the "true" predicate is not gradable, unlike "beautiful"—which can be graduated (p. 126).

For Frege, logic, like ethics, is the normative science based on the most general laws of truth (p. 128). Next, he wrote:

> Logic is concerned with the laws of truth, not with the laws of holding something to be true, not with the question of how men think, but with the question of how they must think if they are not to miss the truth. (p. 149)

That is why the laws of truth are contrasted with the laws of thinking and the laws of judging that psychology deals with (p. 145–146). Moreover, "[t]he laws of truth, like all thoughts, are always true if they are true at all" (p. 148).

In the unfinished textbook on logic, Frege clearly wrote about the indefinability of truth for the first time: "Truth is obviously something so primitive and simple that it is not possible to reduce it to anything still simpler" (p. 129). Therefore, he considered truth to be indefinable. In such cases one only has to "[...] to lead the reader or hearer, by means of hints, to understand the word as it is intended" (Frege, 1892/1990a, p. 183). In a later paper, from 1914, this activity would be called elucidation

[*Erläuterung*], distinguished from defining (Frege, 1914/1983, p. 207).[13] However, it seems strange because he described elucidation as a pre-scientific activity, beyond science, it was only its propedeutics.

Frege devoted a lot of space to the concept of thought, to which the predicate "true" fundamentally referred. Next, the predicate "true" referred to a declarative sentence.

In his concept of thought, first, truth (or falsity) is not a matter of recognition by one person or another. The objectivity of truth (or falsity) results from the "fixing" in objective thought. Frege believed that this guarantees objectivity in science. He wrote:

> [...] thoughts have [not] to be thought by us in order to be true. [...] Thoughts are independent of our thinking. A thought does not belong specially to the person who thinks it, as an idea does to the person who has it. [...] A contradiction between the assertion [*Behauptungen*] of different people would be impossible. (Frege 1897/1983, p. 127)

Next, thought is not mental. But if it were, then:

1. "[...] its truth could only consist in a relation to something external, and that this relation obtained would be a thought into the truth of which we could inquire" (p. 127).
2. Mathematical propositions would look as follows: "It has been observed that with many people certain ideas form themselves in association with the sentence '2 + 3 = 5'" (p. 134).

To sum up, I would like to emphasise the similarity between the unpublished *Logik* (1897) and *Der Gedanke* (1918–1919) published twenty years later; however, this topic needs further study.

Truth in Particular Contexts

Problem of Antinomy

Questions of truth, falsity and words connected with them occurred in the Frege—Russell correspondence. This exchange of letters referred

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[13] In this paper, *Erläuterung* is translated as illustrative examples, however, in the literature in this context *elucidation* is used. See (Weiner, 2002; Besler, 2010, p. 148–149).

mainly to the problem of antinomy and the edition of the second volume of *Grundgesetze der Arithmetik* (Frege, 1903/2009a, with Frege, 1903/2009b).[14] Truth appears there in philosophical and logical contexts.

Frege tried to convince Russell on his philosophy of language, in which task he failed. Nevertheless, we have many clear passages related to philosophical solutions adopted for truth and falsity. They do not bring anything new to Frege's previous point of view, but they are worth special attention due to their precision and the unambiguity of the wording. Here are two examples, one from 1902, the other one written a year later:

> As you know I distinguish between the sense and the meaning [*Bedeutung*] of a sign, and I call the sense of a proposition [*Satz*] a thought and its meaning a truth-value. All true propositions have the same meaning: the true; and all false propositions have the same meaning: the false. (Frege, 1902/1983c, p. 149)

> [...] all propositions that express a true thought mean the same, and likewise all propositions that express a false thought. We have, e.g., $3 > 2. \supset .2^2 = 4$ and $2^2 = 4. \supset. 3 > 2$; consequently: $3 > 2. = .2^2 = 4$. (Frege, 1903/1976, p. 158)

The task of the True and the False in Frege's logic is shown by the quotation from yet another letter: "Regarding the last points you touch on, I shall make the following: $\acute{\varepsilon}(—\varepsilon)$ is a class comprising only a single object, namely the true, and $\dot{\varepsilon}(\varepsilon = \top\overset{\mathfrak{a}}{\smile}\mathfrak{a} = \mathfrak{a})$ is a class comprising only a single object, namely the false" (Frege, 1902/1976b, p. 137).

Truth is essentially connected with the Law V, leading to antinomy. This law says: the equality of the value-ranges of two functions is equal to the general equality of those functions for every argument, in Frege's notation (1893/2009, p. 36):

$$\vdash(\acute{\varepsilon}f(\varepsilon)=\acute{\alpha}g(\alpha))=(-\overset{\mathfrak{a}}{\smile}-f(\mathfrak{a})=g(\mathfrak{a}))$$

Frege tried to save his system of logic against antinomy. In an afterword to the second volume of *Grundgesetze der Arithmetik*, he introduced a limitation of the generality of a function in defining the basic concepts of the arithmetic of natural numbers (Frege, 1903/2009c). The expressions the True and the False appeared there only once:

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[14] For more on the Frege—Russell correspondence see (Besler, 2016).

> [...] the extension of a concept under which only the True falls should be the True and that the extension of a concept under which only the False falls should be the False. These determinations suffer no alteration under the new conception of the extension of a concept. (Frege, 1903/2009c, p. 263)

I would like to add that Frege was aware of a problematic aspect of the Law V ten years before Russell's discovery. He wrote:

> If anyone should believe that there is some fault, then he must be able to state precisely where, in his view, the error lies: with the basic laws, with the definitions, or with the rules or a specific application of them. If everything is considered to be in good order, one thereby knows precisely the grounds on which each individual theorem rests. As far as I can see, a dispute can arise only concerning my basic law of value-ranges (V), which perhaps has not yet been explicitly formulated by logicians although one thinks in accordance with it if, e.g., one speaks of an extension of a concept. I take it to be purely logical. At any rate, the place is hereby marked where there has to be a decision. (Frege, 1893/2009, p. VII)

It could be said that Frege doubted the truth of the Law V from the beginning (comp. Heck, 2010, p. 349–352), and unfortunately the Law V was crucial for his logistic program.

Geometry

Frege became acquainted with a new approach to geometry, which was David Hilbert's *Grundlagen der Geometrie* (1899). He was very impressed with this book, however, he could not agree with Hilbert. Frege did not accept (or did not understand) geometry understood as a formal system, allowing many models, including models of Euclidean geometry.

The topic of truth appears in the Frege—Hilbert correspondence in the context of different understanding of axioms in the system of geometry and their tasks. For Frege, axioms are true propositions. They do not need any proving, because "[...] our knowledge on them flows from a source very different from the logical source, a source which might be called spatial intuition. From the truth of the axioms it follows that they do not contradict one another" (Frege, 1899/1976, p. 37).

Frege assumed that all axioms of Euclidean geometry were irrefutable, he was convinced that "[...] it will be impossible to give such an example in the domain of elementary Euclidean geometry because all the axioms are true in this domain" (Frege, 1900/1976, p. 71). Moreover, according to Frege, the axioms were necessarily consistent with each other. The truth and consistency of the axioms mutually conditioned each other.

Hilbert did not accept Frege's (idealistic) understanding of thought. Therefore, the philosophical background, always present in Frege's analysis, and rarely found in letters addressed to him, significantly disunited these correspondents. According to Frege, the thoughts-axioms are expressed in sentences-axioms (Blanchette, 2015, p. 111).

Similarly to many topics developed by Frege, the philosophy of language also appeared in the context of truth in geometry. In an unpublished paper on geometry from 1899–1906 he wrote:

> In the majority of cases what concerns us about thought is whether it is true [*Wahrsein*]. The most appropriate name for a true thought is a truth [*Wahrheit*]. A science is a system of truths [*Wahrheiten*]. A thought, once grasped, keeps pressing us for an answer to the question whether it is true [*Wahrsein*]. We declare our recognition of the truth of a thought, or as we may also say, our recognition of the truth [*Wahrheit*], by uttering a sentence with assertoric force. (Frege, 1899–1906/1983, p. 168)

After completing the correspondence with Hilbert, Frege returned several times to expressing his opinion on Hilbert's new approach to geometry. In one of the published articles, he repeated the thesis about the truth and consistency of axioms (Frege, 1903/1990).

It is worth pointing out at this juncture that Frege's comments on Hilbert's geometry were widely discussed, and Frege went down in the history of geometry as a defender of the truth of axioms (Freudenthal, 1957/2009, p. 494).

Expression generality

Frege combined truth with the expression of generality. In his logic, the generality of expressions is written in two ways:

1. Using the quantifier symbol.
2. By the appropriate type of variables.

As we may see, only the universal quantifier

$$,\text{—}\overset{\mathfrak{a}}{\smile}\text{—}\,\Phi(\mathfrak{a})'$$

is introduced as a separate symbol. It was characterized in relation to truth-values as I have already written in this article.

Frege also used an existential quantifier (not calling it such), although it was absent as a separate sign. It was written with the use of the universal quantifier and negation-stroke, for example:

$$\vdash\!\!\neg\!\overset{\mathfrak{a}}{\smile}\!\neg\; \mathfrak{a}^2=1\,,$$

and read as "there is [*es gibt*] at least one square root of 1" (Frege, 1893/2009, p. 12). Probably due to the lack of a separate symbol for the existential quantifier, Frege connected truth only with the expression of generality and did not refer truth to the existential quantification.

Frege also signified generality by using an appropriate variable letter. For objects, they are letters of the Latin alphabet, *x*, *a*, etc. (Frege, 1893/2009, p. 11). He wrote:

> In order to obtain an expression for generality, one might have the idea of defining: "Let us understand '$\Phi(x)$' as the True if the value of the function $\Phi(\xi)$ is True for every argument; otherwise it shall refer to the False". (Frege, 1893/2009, p. 11)

For functions, they are capital letters of the Greek alphabet, Φ, Γ, etc. (Frege, 1893/2009, p. 35). There are also appropriate symbols for arguments of functions of higher degrees (Frege, 1893/2009, p. 60–61).

Truth is, for Frege, substantially connected with expressions of generality because, when assuming the above rules, they express the true thoughts (Frege, 1898–1903/1983, p. 162).

In this context, the assertion-stroke also appeared. Only sentences or formulas with a specific general domain can be preceded by the assertion-stroke, that is to say formulas with a quantifier or variables that express generality.

An example of a formula substantially connected with generality is the Law V, it expresses that equality of the value-ranges of two functions is equal to the general equality of those functions for every argument (Frege, 1893/2009, p. 36). After discovering that it leads to antinomy (Frege, 1902/1976a), Frege limited the domain of the functions, and in doing so he limited the scope of truth of the Law V (Frege, 1903/2009b, p. 262–263). At this juncture, as Frege saw it, he lost the generality of arithmetical propositions (Frege, 1903/2009b, p. 255).

Other Contexts

In the period examined here, the topic of truth also appears in other contexts, presented in Frege's correspondence with the great mathematicians of those times: David Hilbert (1862–1943) and Giuseppe Peano (1858–1932).

Frege's first letter to Hilbert concerned mathematical symbolism, and in this context the subject of truth appeared. Frege referred to mathematics understood as a game of symbols, in isolation from their references and wrote:

> A mere mechanical operation with formulas is dangerous (1) for the truth of the result and (2) for the fruitfulness of the science. The first danger can probably be avoided almost entirely by making the system of signs logically perfect. As far as the second danger is concerned, science would come to a standstill if the mechanism of formulas were to become so rampant as to stifle all thought. (Frege, 1895/1976, p. 33)[15]

Another thread comes from Frege's letter to Peano (undated, but surely written between 1896 and 1903) and concerns consequences resulting from various ways of defining equality in arithmetic, as a result of which

> [...] mathematicians agree indeed on the external form of their propositions but not on the thoughts they attach to them, and these are surely what is essential. What one mathematician proves is not the same as what another understands by the same sign. We only seem to have a large common store of mathematical truths [*Wahrheiten*], This is surely an intolerable situation, which must be ended as quickly as possible. (Frege, 1896–1903/1976, p. 126)

In these circumstances, Frege proposed, first of all, accepting identity, "complete coincidence" as the reference of the equality-sign (Frege, 1896–1903/1976, p. 126). Furthermore, thanks to distinguishing the equality at the level of sense from the equality at the level of reference, mathematics will be protected from generating always true, but boring instances of the principle of identity, $a = a$ (Frege, 1896–1903/1976, p. 126).

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[15] In this letter Frege referred to his article (Frege, 1885/1900). It is not clear if Hilbert knew this paper.

Later Utterances on the Truth

Shortly after the publication of the position (1903b/2009) closing the period examined in this article, Frege introduced important new points to his theory of truth. It remains a separate topic as to how the problem of antinomy conditioned these changes, however, this topic requires a separate careful study.

Frege's penultimate letter to Russell is worth attention. There, Frege once again emphasised the particularity of the "true" predicate. What is more, there appears—for the first time—an excerpt, which can be treated as a basis for crediting Frege with a redundant understanding of truth:

> [...] "true" is not a predicate like "green". For at bottom, the proposition "It is true that 2+3=5" says no more than the proposition "2+3=5". Truth is not a component part of a thought, just as Mont Blanc with its snowfields is not itself a component part of the thought that Mont Blanc is more than 4000 high. (Frege, 1904/1976, p. 163)

Whether Frege actually assumed the redundant theory of truth and, if so, to what extent[16] is beyond the scope of this paper.

In the above-quoted letter there is also a clearly formulated principle of extensionality, referring to the substitutability *salva veritate* of expressions. This principle had already been used by Frege in his letter to Russell. He gave the following example of two propositions referring to the True: $2 + 3 = 5$ and $2 = 2$. Therefore it is correct to write: $(2 + 3 = 5) = (2 = 2)$ (Frege, 1893/2009, p. 9), where the sign "=" between the brackets shows the identity of the expression in brackets on the level of reference, but not on the level of sense. In the letter to Russell there is the following wording of the principle of extensionality used here:

> Then and only then does the meaning of the proposition enter into our considerations; it must therefore be most intimately connected with its truth. Indirect speech must here be disregarded. Disregarding it, we can therefore say that true proposition can be replaced by any true proposition without detriment to its truth, and likewise any false proposition by any false proposition. (Frege, 1904/1976, p. 165)

[16] For example Baldwin maintained that Frege did not assume the deflationary theory of truth (Baldwin, 1997, p. 9).

In 1918, Frege received from Ludwig Wittgenstein (1889–1951) a manuscript of *Logisch-philosophische Abhandlung* (Wittgenstein, 1921/1997) and inspired by it he published an article (Frege, 1918–1919/1990a). There he repeated many of his theses from the earlier paper (Frege, 1897/1983), adding arguments against the correspondence theory of truth[17] and the expression "the realm of thought" and its philosophical description.

Conclusion

For Frege, during the period of publishing the two volumes of *Grundgesetze der Arithmetik*, truth was an important category in the fields of philosophy of language (he starts from this aspect), formal logic (here truth plays the role of the key "tool"), philosophy of logic (expression of generality and the problem of antinomy), philosophy of mathematics (the problem of true axioms in geometry and understanding of equality in arithmetic) and ontology (idealistic understanding of the realm of thought, which is really connected with the truth). Bearing in mind the development of Frege's views on truth (Sluga, 2002), here I collect his main theses from the investigated period:

1. At the starting point, truth is examined on the basis of an ordinary language.
2. In logic, truth is expressed by the assertion-sign, therefore, truth is connected with judging.
3. Truth refers to logic more than any other science.
4. Truth is a normative category, because logical laws—as true—determine the direction of thinking.
5. Language expressions have sense and reference, the reference of sentences is truth-value (the True or the False).
6. Logic connectives, quantifier and logical entailment are characterized by truth-values.
7. The truth-bearers are: first thought, then proposition (or sentence) and language of science. Never ideas [*Vortsellungen*].

[17] Frege's criticism of the correspondence theory of truth was presented in (Sluga, 2007, p. 4–9; Baldwin, 1997).

8. The philosophical notion of thought is essentially connected with truth, as a certain unchanging objective ideal reality.
9. Truth is a primitive, indefinable term.
10. Truth—next to consistency—is an important notion describing geometrical axioms.

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