Studia Semiotyczne, t. XXXII, nr 2 (2018), s. 153–185
ISSN 0137-6608, e-ISSN 2544-073X
DOI: 10.26333/sts.xxxii2.09

Projekty i dyskusje

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DOES SCIENCE PROGRESS TOWARDS EVER HIGHER SOLVABILITY THROUGH FEEDBACKS BETWEEN INSIGHTS AND ROUTINES?

Summary: The affirmative answer to the title question is justified in two ways: logical and empirical. (1) The logical justification is due to Gödel's discovery (1931) that in any axiomatic formalized theory, having at least the expressive power of PA (Peano Arithmetic), at any stage of development there must appear unsolvable problems. However, some of them become solvable in a further development of the theory in question, owing to subsequent investigations. These lead to new concepts, expressed with additional axioms or rules. Owing to the so-amplified axiomatic basis, new routine procedures like algorithms, can be reached. Those, in turn, help to gain new insights which lead to still more powerful axioms, and consequently again to ampler algorithmic resources. Thus scientific progress proceeds to an ever higher scope of solvability. (2) The existence of such feedback cycles – in a formal way rendered with Turing's systems of logic based on ordinal (1939) – gets empirically supported by the history of mathematics and other exact sciences. An instructive instance of such a process is found in the history of the number zero. Without that insight due to some ancient Hindu mathematicians there could not arise such an axiomatic theory as PA. It defines the algorithms of arithmetical operations, which in turn help intuitions; those, in turn, give rise to algorithmic routines, not available in any of the previous phases of the process in question. While the logical substantiation of the point of this essay is a well-established result of logico-semantic inquiries, its empirical

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claim, based on historical evidences, remains open for discussion. Hence the author's intention to address philosophers and historians of science, and to encourage their critical responses.

KEY WORDS: Algorithm, arithmetic, axiom, axiomatic formalized theory, concept, decidability, feedback, insight (intuition), mathematics, mechanism, mentalism, oracle, problem, problem-solving, progress, routine procedure, science, solvability.

INTRODUCTION: ON THE DEBATE BETWEEN EXTREME MECHANISM AND BALANCED MENTALISM

Albert Einstein is reported to have once said: "if you gave me an hour to solve a problem, I would use the first 55 minutes to consider if it is the right problem."

0.1. The existence of the hot debate of a problem as a warrant of its non-triviality

Does this essay's title express the right problem? The anecdote does not say what criteria of being the right problem Einstein might have in mind. However, it does not seem risky to assume the following conditions. (1) The solution is not trivial, i.e., it requires research. (2) The problem in question is likely to be solved with the available means of research.

That our title question is not trivial is evident from its being the focus of the unsettled debate between two opposing approaches to Artificial Intelligence systems.

One of them claims the following. An AI system is able to imitate natural human intelligence, as in some important cases it produces identical solutions, but in a way essentially different from that characteristic of the human mind. To wit, AI systems proceed in a purely routine way, while the typically human process of problem-solving consists in mutual interactions between creative insights and programmed routines. This approach deserves to be named *balanced mentalism*, as focussing on a balance between mental insights and robot-like routines. Insights, that is, c r e a t i v e c o n c e p t u a l i z a t i o n s, are the source of routines, and those, in turn, facilitate new creative conceptualizations (cp. **5.2**).

To evidence such a feedback, as well as the nature of conceptual creativity, let us consider the discovery of the number zero by Hindu mathematicians, more than a thousand years ago. The creation of this concept was a deep and penetrative insight into the realm of arithmetic, leading to the positional notation of numbers. This story nicely exemplifies the *feedback* between insights and routines, that is, mechanical procedures in performing various tasks: in particular, problem-solving. Owing to the positional notation based on zero, it was possible to create algorithms for arithmetical operations: addition, multiplication, etc.; a special role is played by binary notation, necessary for the functioning of digital electronic computers.

According to the opposite approach, the way of producing a requested solution is exactly like that in the case of natural human intelligence. To wit, in both cases the process of problem-solving is a mechanical (i.e., algorithmic) routine. Humans are proud of their creativity, but in fact that alleged creativity is a mechanical process, whose mechanism is still unknown. However, it should be discovered owing to future, more advanced, inquiries into the enormous complexity of the human brain. Let this approach be called *extreme mechanism*.

The said adjectives will be, for brevity, omitted in what follows, if not needed in the given context. Now we have a hint that our title question is not trivial, since it reflects a real mind-philosophical controversy between mentalism and mechanism.

To define both options in more detail, let us notice the following. Mechanism is the claim that (i) the human mind is identical with the human brain, and (ii) the latter is equivalent to the Universal Turing Machine (UMT) as defined in Turing's study (1936). In both cases – it is assumed – any successful problem solving is an a l g o r i t h m i c p r o c e d u r e; it is convenient to call it a *routine*. The extremity of this kind of mechanism consists in its being extremely categorical; the adjective stresses the fact that followers of mechanism regard their view as unreservedly rational and scientific, and without any compromise with mentalism.

Mentalism claims that at least one member of the above conjunction has to be false. This follows from the fact that the extent of fitting and fruitful solutions produced by the human mind is essentially ampler than that possible to UMT. Mentalism would be extreme if asserted that mental acts alone suffice to solve logical and mathematical problems, without any need of resorting to routines, i.e., purely mechanical procedures; the admittance of such procedures makes it a moderate approach.

0.2. On the odds that the mechanism-mentalism contention can be solved in the present state of science

Let us dwell a while on the second feature of the right problem, that of solvability: the problem in question should be likely to be solved with the available means of research. This postulate is relevant to the argumentative strategy adopted in this essay. The essay is meant as a contribution to the controversy between mechanism and mentalism, a contribution based on some metamathematical results. In this respect it is like Webb's book (1980) which bears the informative title *Mechanism, Mentalism, and Metamathematics*. I make use of the same terms to express the opposite stance, to wit, the balanced mentalism.

It is not possible to present Webb's extensive argumentation, and offer convincing counterarguments, within the size limits of the present paper. Instead, I focus on some possible philosophical and semantical foundations of mechanism which preceded in time the metamathematical problems and results of Hilbert (1928), Gödel (1931, 1936), and Turing (1936, 1939).

Moreover, I regard it as pertinent to preserve a kind of symmetry, to wit, to consider as an opponent of mentalism somebody as renowned and influential as the three mentioned authors, and, like them, engaged in the issues of philosophy of mathematics and logic, philosophy of mind, and semantics.

Having this in mind, the best possible choice seems to be Ludwig Wittgenstein as the author of *Tractatus Logico-Philosophicus* (1922) considered in the context of related views, such as Russell's logical atomism and the Vienna Circles project of unified science (supported by mechanist assumptions).

The sequence of discussion is motivated by the fact that Gödel's balanced mentalism, corresponding in a way with Hilbert's and Turing's problems, requires a more extensive presentation than the aphoristic utterances of Wittgenstein. With the conceptual apparatus of Gödel and Turing, e.g. the concept of decidability, it is easier to interpret some of Wittgenstein's maxims which, without such an aid, sound rather mysterious.

It is in order to briefly explain the title phrase "ever higher solvability". It is to mean the following. Within a definite period of time, at every stage of scientific development the means and methods of problem-solving become more numerous and exact than at any preceding stage. The proviso "within a definite period of time" follows from our knowledge about some past periods. For instance, in physics, the beginning of the period of increasing solvability may be dated only after the entrance on the scene of history of Copernicus and Galileo. In logic, the intense increase of solvability starts from the axiomatic and formalized system of logic rendered in a precise symbolic language by Frege (1879).

The notion of higher solvability can be applied as well to the solvability of the contention between mechanism and mentalism. According to the present author, after Gödel's and Turing's discoveries, the degree of solvability is higher than in the period of Wittgenstein, and proves to be in favour of balanced mentalism. However, there are authors who claim that recent results in Artificial Intelligence, neurology, robotics, etc. yield evidence in favour of mechanism, even the extreme. Weak points, if there are any, in Wittgenstein's mechanist stance, do not seem to them relevant in the present advanced state of scientific knowledge. If they are ready to defend mechanism on that or another basis, such a rejoinder will be welcomed by the present defender of balanced mentalism.

1. Gödel's dynamic vision of ever advancing frontiers of solvability

1.1. Gödel's incompleteness theorem in the light of the opposition: "frontier" vs "limit"

There is in English a suggestive distinction between the concepts of *frontier* and *limit*. Though fairly subtle, it is thought-provoking, and crucial for the present discussion. To realize its role, let us look into dictionaries.

Limit: the point, edge, or line beyond which something ends, may not go, or is not allowed.
Frontier: 1. A region just beyond or at the edge of a settled area. 2. An unexploited so far area for discovery or research.

In the latter definition, the second meaning belongs to the vocabulary of academic communities, while the first, which gave origin to the second, is taken from the idiom of the American pioneers. These with their drive, bravely overcame the limits of their hitherto exploited lands, pushing the frontiers of their estates more and more to the West.

This linguistic phenomenon in English lexis helps us to more precisely render the philosophical significance of Gödel's incompleteness theorem. People are accustomed to saying that Gödel discovered the *limits* of solvability of mathematical theories; but, instead, it should be said that he discovered the *frontiers* of solvability. That is to say, he drew the critical line to mark a limit of algorithmic procedures, and devised the strategy of its overcoming in the march towards new lands of mathematical truths.

New entities are first grasped through wordless insights, then named, and defined in axiomatic manner. If the theory in question is not only axiomatized (like Euclid's geometry), but also formalized (like Hilbert's geometry), then we can obtain algorithms for automated proving (provers) and for automated checking of handmade proofs (checkers).

As new concepts are introduced into a theory, and then axiomatized and formalized, the scope of its algorithmic solvability becomes ampler. Gödel (1936) gave a classic example of such a process. His approach consists in considering an infinite sequence of arithmetical theories – such that for every theory there is a theory having a greater scope of solvability, due to some conceptual innovations, to wit, introducing more and more abstract concepts of set.

Before addressing this approach in greater detail, it is in order to consider the method of attaining new concepts. It is the attainment which does not need any resort to the concepts which already exist in the given theory.

In the following subsection **1.2**, after mentioning the axiomatic method of introducing new ideas, we are to deal with the axiom of comprehension to introduce the notion of *abstract set*. This concept is presupposed in Gödel's example of amplifying the range of solvability, and pushing the frontiers of mathematics. This is why we ought to devote special attention to that formula.

1.2. On the idea of set as introduced by the axiom of comprehension

There is a key difference between introducing new expressions and introducing new concepts. The former is not bound to create a conceptual novelty; a new expression may express with new words an old concept, one that already exists in our language. In such a case, we introduce the new expression for the sake of convenience, as shorter, having desirable associations, etc. Such a job is done by *normal definitions*. They should satisfy the conditions of eliminability and of non-creativity; this implies that there is no increase in the amount of information.

In order to introduce a new concept, i.e., one carrying new information, and so giving us a chance of solving problems hitherto insolvable, we use *meaning postulates*. These are creative, to enable answering questions, hitherto unanswerable. There are two kinds of meaning postulates: operational definitions in empirical theories, and axiomatic definitions in deductive theories.[1]

At the very start of the discourse on sets, it is in order to explain that the term *set* will be here used interchangeably with *class*, as meaning exactly the same. In some systems of set theory these two forms are employed to distinguish two kinds of multitudes. That practice is justified by some theoretical needs, but in the present discussion such sophistication is not needed; the use of one or the other name will be motivated purely by stylistic convenience.

The formula to introduce the concept of set is standardly called the axiom of *comprehension*. Some other names are also in use. Among them *axiom of abstraction*. This is even more telling than the standard term, since it defines the abstract concept of set, and occurs in the phrases "abstraction class" and "definition by abstraction". However, to make referring to literature easier, I am to follow the standard version, la-

[1] More on this subject: Marciszewski (1981), see especially "Definition" pp. 86–96, Sections 2.2ff, 4.3 and 5.3.

belling it "UC" for "Unrestricted Comprehension" (the sense of the adjective – explained below).

The axiom reads: *There exists a set y whose members (represented by x) are precisely those objects that satisfy the condition φ.*

Strictly speaking, it is not a concrete single axiom, but what is called "axiom schema" since φ represents infinitely many formulas which could be substituted for this variable. In symbols the axiom reads as follows:

UC: $\forall x \exists y\, (x \in y \Leftrightarrow \varphi(x))$

The set whose existence is so postulated is named an *abstraction class*. In logical notation it reads $\{x: \varphi(x)\}$ to mean: the class of entities which satisfy the condition $\varphi(x)$.

UC was used, in the early days of mathematical logic, as the basis of "naive" set theory, that is, the one being developed before a strict axiomatization has been devised. This formula should be used cautiously since in some substitutions for φ it leads to Russell's paradox. To avoid that antinomy, relevant restrictions have been added. In that restricted form the axiom entered the axiomatic set theory of Zermelo and Fraenkel, labelled ZF, or ZFC ("C" for axiom of Choice, if added to ZF).

In order to distinguish the original simple version from that restricted one adopted in ZFC, the former has been called the "unrestricted comprehension axiom". In the present discourse those restrictions are not necessary, hence it is UC which will be referred to.[2]

1.3. The increase of the scope of solvability owing to the axiom of comprehension

The axiom schema UC can produce an infinite sequence of sets of ever higher order. The greater the order of a language, the higher the degree of solvability possessed by theories expressible in that language.

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[2] See https://en.wikipedia.org/wiki/Axiom_schema_of_specification. This article is an extensive account on the forms and history of this axiom. There is in it the thought-provoking remark that the remaining ZFC axioms became necessary to make up for some of what was lost by changing the axiom schema of unrestricted comprehension into the restricted one (called also the axiom of specification).

The order of a set, likewise the order of the respective language, is characterized as follows.

Let *x* range over individuals, labelled as entities of order zero, and y range over sets-of-individuals, hence entities of order one.

$\mathbf{UC_1}$: $\forall x \exists y\, (x \in y \Leftrightarrow \varphi(x))$

Now, let *x* range over sets-of-individuals (entities of order one), and *y* over classes-of-sets-of-individuals, these classes being entities of order two.

$\mathbf{UC_2}$: $\forall x \exists y\, (x \in y \Leftrightarrow \psi(x))$

Next comes $\mathbf{UC_3}$ to define sets of the third order: the class-of-classes-of-sets-of-individuals. And so on, up to infinity.

The notion of set as defined by $\mathbf{UC_1}$ is a conceptual innovation with respect to the notion of an *individual*. Another notion of set, that defined by $\mathbf{UC_2}$ is a conceptual innovation with respect to the notion of a *set of individuals*, etc., so we deal with an infinity of conceptual innovations.

This results in another infinity of conceptual innovations, to wit, infinitely many concepts of the quantifier. The symbol $\forall x$ when in $\mathbf{UC_1}$ it ranges over individuals means something other than in the case of ranging over sets of individuals, as in $\mathbf{UC_2}$. And so on.

Such an ordering of sets and quantifiers determines the order of languages. The sentence "this is my pair of shoes" belongs to second-order English, for the term "pair" denotes a set. The saying "in this store we have a set of 100 pairs of black shoes" belongs to third-order English.[3]

The language of a logical theory which contains only the quantifiers of first order, is said to be the first-order language. In the case of quantifiers of at most second order, we are dealing with a second-order language, and so on.

[3] This should be considered by nominalists who try to discourage us from any confidence in set theory; let they try to get rid of higher-order phrases in ordinary languages.

Thus, being familiar with the infinite ladder of orders, we are ready to appreciate the significance of Gödel's statement in his *communiqué Über die Länge von Beweisen* (1936), that is, *On the length of proofs*. The main point runs as follows (translated from the German):

> Thus, passing to the logic of the next higher order has the effect, not only of making provable certain propositions that were not provable before, but also of making it possible to shorten, by an extraordinary amount, infinitely many of the proofs already available (Gödel, 1986, p. 397).

This assertion was neither demonstrated nor exemplified by Gödel himself. This was done by other authors some years later.[4]

1.4. The endless evolution of mathematics having its source in the inexhaustibility of the world of sets (Gibbs Lecture)

The unbounded openness of mathematical language, exemplified by Gödel (1936) with the case of number theory, was considered by him more extensively a dozen years later in the famous *Gibbs Lecture*. In this case, the idea of infinite sequences of axiomatic systems, was extended onto the problem of axiomatizing set theory.

> If one attacks this problem, the result is quite different from what one would have expected. Instead of ending up with a finite number of axioms, as in geometry, one is faced with an infinite series of axioms, which can be extended further and further, without any end and, apparently, without any possibility of composing all these axioms in a finite rule producing them. You will realize, I think, that we are still not at the end, nor can there ever be an end to this procedure of forming the axioms, because the very formulation of the axioms up to a certain stage gives rise to the next axiom (Gödel, 1995, pp. 306–7).

The existence of such infinite chains of axiomatic systems, considered both in the paper (1936) and in the Gibbs Lecture, Gödel inferred from the *principle of inexhaustibility of objective mathematics*. And that, in turn, he conceived as a consequence of his incompleteness theorem. It entails the existence of an infinite domain of mathematical facts which cannot be matched by the set of axioms at any stage

[4] S. R. Buss (1994) produced a detailed proof, while George Boolos offered a nice exemplification in the seminal study *A Curious Inference* (1987). A comment on Boolos's contribution in found in Marciszewski (2006).

of development of mathematical knowledge, called by him *subjective mathematics*.

It should be noted that there is a difference between the endless producing of new axioms in set theory and in number theory. Let us consider Gödel's saying about set-theoretical axioms, that they grow to infinity "apparently, without any possibility of composing all these axioms in a finite rule producing them" (see the quotation above).

In the case of number theory, as treated by Gödel (1936), there is an obvious way of obtaining new systems of axioms: to add quantifiers of the next higher order to those already existing. This is a simple finite rule of producing an ordered sequence converging to infinity.

This makes a difference from set theory as to the kind and degree of conceptual inventiveness. In the case of order degrees in number theory, it is enough to have an intuition concerning the existence of set orders, as entailed by the axiom schema of comprehension. This intuition gives rise to the rule of obtaining new axioms of ever higher order. These new ones are, in a sense, not innovative: each next element of the sequence is new, but produced according to the same general instruction.

With the theory of sets – says Gödel – it is different. Any progress in winning its more powerful axiomatization requires a new insight. For instance, Gödel hoped that in the future indubitable axioms would be found to decide the continuum hypothesis, owing to the deeper intuitions likely to arise in the meantime.

However, inventive insights are no infallible revelations. In the progress of science it is necessary to make steps forward, but not always are they steps in the intended direction. If not, a step backwards may prove necessary in order to look for a better solution: even in mathematics, as was emphasized by Gödel in the question about the possible future fate of the continuum hypothesis.

Such a vision of science – sometimes erring, never ending, and ever marching forward – is characteristic of our current philosophy of science. To get a deeper understanding of this new landscape, let us have a look into a time in which the nature of science was conceived in a way deeply different from that of ours.

2. The farewell of modern science to the models of omniscient daemons

2.1. A kinship between the daemons of Laplace and Hilbert. The rise and fall of their careers

Everybody must have heard of the daemon imagined by Pierre Laplace (1749–1827), while nobody speaks of a similar entity considered by David Hilbert (1862–1943). Nevertheless, Hilbert's daemon does exist as a hypothetical imaginary being. These two daemons are alike in their rise and fall.[5]

Laplace's determinism in physics is personified by the omniscient daemon. Suppose, he knows the precise location, momentum and history of every atom in the universe. Then he can compute, on the basis of classical mechanics, the past and the future of the whole universe.

Hilbert entertained the notion of a universal algorithm solving computationally any mathematical problem encountered, thus being like an omniscient daemon in the universe of mathematics. The very term "compute" hints at the kinship of the said concepts. In some recent publications, e.g., in Rukavicka's (2014) paper, one disproves Laplace's demon using Turing machines. Turing, on the same basis (an abstract machine), refuted Hilbert's programme. No wonder, since what both daemons have in common, is cognitive maximalism: each of them can solve each problem concerning his universe: the physical universe in Laplaces's case, and the mathematical in Hilbert's. Here is Laplace's statement about his daemon's problem-solving power.[6]

> We may regard the present state of the universe as the effect of its past and the cause of its future. An intellect which at a certain moment would know all forces that set nature in motion, and all positions of all items of which nature is composed, if this intellect were also vast enough to submit these data to analysis, it would embrace in a single formula the movements of the greatest bodies of the universe and those of the tiniest atom; for such an intellect nothing would be uncertain and the future just like the past would be present before its eyes (Laplace, 1951, p. 4).

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[5] The idea of telling the history of science in terms of daemons is borrowed from Webb (1980).

[6] The quotation which follows is taken from the Wikipedia article "Laplace's demon".

Hilbert's daemon is to be understood as a perfect, infallible mind which is omniscient in the realm of mathematics. This philosophical description is equivalent to the technical formulation of what Hilbert called *Entscheidungsproblem*. It runs as follows:

> **[A]** The decision problem gets solved if one knows a procedure which for a given logical expression allows one to decide, with finitely many steps, about its validity or its satisfiability.
>
> **[B]** The solution of this decision problem has a fundamental impact for all those theories whose statements are capable of being logically derived from finitely many axioms (Hilbert, Ackermann, 1928, p. 73).[7]

The phrase "capable of being logically derived from finitely many axioms" means: *solvable by a mechanical procedure with the use a formalized system of predicate logic*, e.g. the system devised by Hilbert and Ackermann in their textbook. Hence, no intellectual insight, no conceptual inventiveness, is needed. An insight may come or not, while the algorithm in question would always be at hand to bring forth the needed infallible solution.

These two visions of cognitive maximalism, Laplace's and Hilbert's, were anticipated by Leibniz. With Leibniz this vision included the computational solvability even of metaphysical and theological questions. Such an excessive optimism must have proved unrealistic, nevertheless it has been very fertile. Frege's ingenious system of logic was inspired by Leibniz's visions and projects, such as those in his famous text:

> If this is done [i.e., an ideal algorithmic language is devised], whenever controversies arise, there will be no more need for arguing among two philosophers than among two mathematicians. For it will suffice to take the pens into the hand and to sit down by the abacus, saying to each other (and if they wish also to a friend called for help): Let us calculate! (Lenzen, 2004, p. 1)[8].

[7] Translated from German and divided into parts by W. M.

[8] Here is the Latin original of 1684 (cf. Gerhardt, 1890, p. 200), translated by Lenzen (2004, p. 1): "Quo facto, quando orientur controversiae, non magis disputatione opus erit inter duos philosophos, quam inter duos Computistas. Sufficiet enim calamos in manus sumere sedereque ad abacos, et sibi mutuo (accito si placet amico) dicere: calculemus." For an illuminating comment to this text, see Benzmüller (2017).

Frege could succeed, while Leibniz could not. Why? Frege's language of logic was preceded by an intense development of the linguistic practice of mathematicians. In this practice appeared individual variables, quantifiers, and higher-order formulas (e.g. in the principle of complete induction).

Due to Frege's achievements, it was possible for Hilbert and his collaborators to create a perfectly formalized language, liable to mechanization. Just with respect to such a precise language the *Entscheidungsproblem* could be stated, and Turing could address this problem in his study (1936) on computability.

As for the fates of daemons narrated in this story, they were different in the two cases. The downfall of Laplace's daemon was sealed by the achievements of modern physics, including quantum indeterminacy; and, additionally, by the computational arguments mentioned above (Rukavicka, 2014).

As for Hilbert's daemon, the story is more involved. Let us look deeper into its context and significance.

2.2. Why Hilbert's daemon had to fail?

Hilbert's key declaration on *Entscheidungsproblem* concerns solvability in terms of either "yes" or "not" – as answers to the question whether a formula of logic is valid, or whether it is satisfiable. Hilbert (1928) meant here first-order predicate logic. The quoted sentence is found in the section "The decision problem in predicate logic, and its significance", where limitation to the first order is rendered by the title of the chapter in which this section is included; it reads "The First-Order Predicate Logic" – FOL, for short.

In spite of such a limitation, it is said in part B that the solution of the decision problem has a fundamental significance for all those theories whose statements are capable of being formalized, that is, "logically derived from finitely many axioms." This condition is satisfied by all mathematical theories, and even more: by all those theories outside mathematics which can be formalized; even philosophical ones, as dreamt of by Leibniz (see footnote 8 in **2.1**), and attempted by Gödel (Benzmüller, 2013, 2015).

To appreciate the economy of linguistic means in FOL, let us notice that only four primitive logical constants suffice to express all possible formulas of FOL: these can be – as in Frege – the symbols of negation and implication, and one (arbitrary) quantifier; plus, the equality sign. The rest of the logical constants can be defined in terms of these primitives. If the set-theoretical symbol of membership (∈) is added, then in the language based on them one can express all the concepts and theorems of mathematics.

A language enjoying such great expressive power is close, in some degree to Leibniz's vision of *scientia uiversalis* in his text *De scientia universali seu calculo philosophico*. This closeness explains what Leibniz's inspiration meant for Frege; this can be seen in Frege's study (1880–1881).[9]

Hilbert, appreciating the great expressive power of Fregean logic, hoped it would suffice to grant the *deductive completeness* to arithmetic. Its axiomatization by Peano and formalization in Hilbert's manner, provided decidability of logic, should yield a procedure of mechanical checking validity of any arithmetical proof.

The adjective "deductive" in the phrase "deductive completeness" means deduction with the use of strictly formal rules, that is, rules referring only to the physical *form* of expressions, not to their meaning.[10]

Only such "physicalism" can ensure the mechanical character of deduction which would make it possible for a machine to produce algorithmic proofs. Gödel's incompleteness theorem is to the effect that some arithmetical truths cannot by proved in such a mechanical way, hence logic cannot provide a universal algorithm to prove every arithmetical truth, as expected in Hilbert's project. However, this project could be revived in the more modest and realistic form to be discussed in the next section.

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[9] See in Frege (1973) where exact bibliographical data about Leibniz's text are also found.

[10] The nature of formal rules can be better understood against the contrastive background of non-formal rules of proof, referring not to linguistic forms but to mental operations. Such are found, e.g., in Descartes' treatise *Regulae ad directionem ingenii*. See https://en.wikipedia.org/wiki/Rules_for_the_Direction_of_the_Mind.

3. Towards universal logic as an infinite sequence of ever stronger machines

3.1. A computational perspective on developing logic towards ever greater deductive potential

Imagine somebody who plans to build a universal plant to produce all possible commodities which may be wanted presently and at any moment of the future. On the other hand, his colleague, in a more realistic mood, entertains another vision of universality. He will gradually exploit his resources: establishing first plants to produce what is currently needed, and then ones taking into account newly arising demands.

This parable is to reflect two approaches to logic as the universal tool of problem-solving. According to Hilbert, it was to be the classical predicate logic whose deductive potential would be sufficient to meet any problems that may arise in axiomatized formal theories, especially in mathematics. The undecidability of predicate logic, demonstrated by Turing (1936) and Church (1936), puts an end to such expectations. Let us look for an alternative.

The theoretical justification of an alternative has been given by Turing (1939) with the idea of oracles (see **5**). Conceptual insights as to the choice of theories able to function as oracles are owed to other researchers. Such choices are motivated by the quest for logical devices needed in the research in question to solve its specific problems.

Such a strategy of "piecemeal engineering" (to use Popper's phrase) is necessary in the face of Gödel's and Turing's results; in particular, Turing's (1936) discovery that there are uncomputable functions whose values cannot be found by any existing Turing machine. Facing this fact, Turing (1939) considered non-mechanical devices – *oracles* (as he called them), each of them able to find values of a certain uncomputable function.

Following the interpretation given by Newman, Hodges (2013), and Turing himself (see **5.1**), being an oracle can be understood as an ability of having insights which result in new concepts, and those in new axioms. New axioms increase the deductive potential of a theory, and thus solve problems having been hitherto unsolvable.

Let us compare this strategy and its theoretical justification with Hilbert's vision of a universal problem-solver, as quoted in **2.1**. According to Hilbert, the universal machine would be capable of solving problems in any axiomatizable theory in which logical derivation would be performed as a mechanical procedure. At the background of this failed project, Turing's alternative strategy of "piecemeal engineering", seems worth considering. Among its representatives, an eminent role is played by Christoph Benzmüller. He writes the following:

> [What this author proposes] utilises classical higher-order logic (HOL) as a unifying meta-logic in which (the syntax and semantics) of varying other logics can be explicitly modelled and flexibly combined. Off-the-shelf higher-order interactive and automated theorem provers can then be employed to reason about and within the shallowly embedded logics. This way Leibniz vision can (at least partially) be realised (Benzmüller, 2017).

Thus the decisive step towards universal logic consists in overcoming the limits of FOL and passing to higher order logics, in accordance with Gödel's (1936) statement, as quoted above (in **1.3**). Another essential move lies in absorbing modal logics, conditional logics, logics of time and space, provability logics, multivalued logics, and free logics, to name just a few examples (Benzmüller, 2017, sec. 3).

Such might be a list of prospective constituents of a universal logic, as a modern, realistic accomplishment of Leibniz's dream. However, this listing reveals a serious difficulty of the enterprise. Besides FOL, almost each item is debatable from one or another philosophical point of view. If so: what strategy should we adopt in our tending toward the universal logic, and tending as well towards its common acceptance by the world of learning?

The strategy of continuing eternal philosophical debates seems least promising. More encouraging are two other approaches, very different from each other, but in a sense complementary. To wit, (i) an appeal to common sense being expressed in our ordinary language, and (ii) arguing from computational efficiency. This twofold approach has been tried in the literature with respect to the second-order logic.

3.2. Some cases of competition on the issue of solvability between FOL and higher-order logics, and between humans and machines

A discussion leading to a greater appreciation of higher-order logics was initiated by George Boolos (1987) with an article entitled *A Curious Inference*. The inference deals with a theorem of arithmetic whose oddity consists in an ineffable difference between the length of a formalized proof in FOL and a proof in 2nd-order logic. The latter occupies ca. two pages of print, while the former – as Boolos calculated – would require more symbols than the number of elementary particles in the universe.

Commenting on that fact, Boolos remarks that the property of being a higher-order language is omnipresent in our everyday speech, without any possibility of expelling it from the ordinary language. Boolos does not dwell on examples, but it is easy to find some. Consider the statement of the following fact: "In the population of this village there are ten married couples, each having three children."

This message does not presuppose any mysterious metaphysics for which nominalists blame higher-order expressions. It is easy to paraphrase this sentence in a mixed idiom in which set-theoretical terms would be inserted into ordinary language, as in the following utterance: "In the set of classes of the given village inhabitants there is the class of ten married couples, each of them being in the parental relation to three children."

In the present discussion it is a rather auxiliary argument, of the *ad hominem* type, addressed mainly to nominalists. These, e.g. Tadeusz Kotarbiński and his followers, try to defend their position by recourse to ordinary language as representing, according to them, common sense, claiming that its grammar does not surpass the limits of the first-order language. If so, let them try to paraphrase in FOL the above-quoted sentence, in order to eliminate names of sets, as "population", "couple" "ten-element set of pairs", "three children".

From a scientific point of view, more significant is another way of testing the utility of higher-order languages. It is nicely exemplified through an experiment described by Benzmüller and Brown (2007) in their extensive report: *The Curious Inference of Boolos in Mizar and OMEGA*. Both Mizar and OMEGA are proof assistants, called also checkers. That is, computer programs devised to check the correctness of proofs in mathematics (but applicable also in other areas). The former requires proofs written in the language of set theory, the latter – of second-order logic. In either case the printout of the checked proof occupies ca. 50 pages.

It is reasonable to suppose that the difference between the second-order printout and the first-order printout is comparable with the difference between their handmade counterparts. And that, according to Boolos's calculation (concerning his case) is like that between several thousand symbols of the 2nd-order version and more than 10^{86} symbols in the 1st-order version. Boolos estimates that in the latter case there would be more symbols than the number of elementary particles in the visible universe, and that amounts to roughly 10^{86} elements.

The moral to be drawn from such speculations is the following. Attempts to use only FOL in mechanized proofs are doomed to failure like analogous attempts at using FOL in some handmade formalized proofs. The proof discussed by Boolos in its non-formalized form requires several lines, i.e., a small fraction of a page, while Andrew Wiles's (1995) non-formalized proof of Fermat's theorem (see **4**) requires much more than a hundred pages of manuscript.

This makes us aware of the enormous complexity (measured by the length) of Wiles's proof in its present non-formalized garb. Thus, we become faced with the phenomenon of the unimaginably higher complexity of the same proof, were it to be formalized according to the requirements of mechanized processing, either by a prover or by a checker.

Compare this question with the illuminating story of Boolos's (1987) "Curious Inference", as sketched in the present subsection. Boolos's case hints at the rapidly growing length (hence complexity) of a proof, when passing from a non-formalized (intuitive) to a formalized approach. Would such a complexity be tractable with the resources of computer memory and time currently available? This is a challenge to be met by competent researchers, especially those who intensely avail themselves of checkers in creating databases of formalized mathematical theories.

4. Relative solvability, algorithmic and intuitive. Morals to be drawn from the success in proving Fermat's theorem

Let us consider, as the motto of this section, the following statement.

> Formal [i.e., algorithmic] decidability is a concept relative to a given formalization of a mathematical theory, and consequently, the fact that some sentence is undecidable in a formal theory does not give any hint as to whether it is intuitively solvable (Placek, 2013, p. 47).

"Formal theory" is to be conceived as a theory suitable to be subjected to mechanization owing to a programming interface. The notion of relative solvability can be instructively illustrated through the sensational story of the career of Fermat's last theorem stated in 1637. It asserts the following.

F: No three distinct positive integers x, y, z can satisfy the equation: $x^n + y^n = z^n$, if $n > 2$.

This theorem was conjectured by Pierre de Fermat in the margin of a copy of Diophantus' *Arithmetica*; he claimed he had a proof that was too large to fit in the margin (where Fermat used to record his comments). In fact, the finding of a demonstration proved so difficult that in the succeeding centuries, up to the year 1995 in which Andrew Wiles published the solution, great mathematical minds were not able to solve the problem, in spite of intense efforts. Now, when we are fully aware of the historical circumstances, some objective reasons for these failures can be explained.

Wiles's proof resorts to algebraic geometry and number theory in their results and methods so sophisticated that they were not available either to Fermat himself or to the next generations of mathematicians, up to the late 20th century. When one distinguishes the content of mathematics in the 17th and 20th centuries, it becomes evident that solvability must be relative to a certain state of this science. Fermat's problem was insolvable with respect to the mathematics of that former period, and solvable with respect to the latter. This is true in a most general sense of "solvability", covering its intuitive and its formal, or algorithmic, varieties.

When stating the *Entscheidungsproblem* (see **2.1**), Hilbert thought of an algorithmic solvability. To wit, the kind realized in a formalized proof liable to be mechanically (automatically) checked or mechanically produced. This interpretation of solvability is confirmed in part B of the decision problem, where Hilbert assigns the attribute of so-interpreted solvability only to "those theories whose statements are capable of being logically derived from finitely many axioms", hence those which are axiomatized and formalized; "logical derivability" in the sense of logic developed by Hilbert, amounts to formalization of proofs.

These ascertainments lead to the question: what kind of solvability characterizes Wiles's solution of Fermat's problem? Certainly it is relative to the state of mathematics in the nineties of the 20th century. Previously the problem was not likely to be solved, even by the most gifted mathematicians, for the lack of relevant concepts and theorems. Has the solution nowadays obtained any chance to be classified as algorithmic? To answer this question, one should realize the size of Wiles's (1995) proof: much more than 100 pages.[11]

There would be two possible ways of getting an answer as to the chance of algorithmic solvability: by the use of a checker or of a prover. Either would require obtaining unimaginably sophisticated software, the next step after giving the proof in question a formalized structure. This would require a vast library, in which all mathematical theories relevant to the proof would be found in a formalized form.[12]

[11] To gain a more professional knowledge about Wiles's result, the Reader is advised to consult the following Internet sources. 1) "The proof of Fermat's Last Theorem" – a fully professional textbook by prof. Nigel Boston (Department of Mathematics University of Wisconsin – Madison); 2) "Wiles's proof of Fermat's Last Theorem" – a much more popular and much shorter article in Wikipedia.

[12] An example of such a device is Mizar Mathematical Library – MML – fruitfully explored by mathematicians in various academic communities. This Library contained in 2017 almost 6000 articles, that is, formalized proofs concerning 36 mathematical theories, submitted by almost 300 authors from 18 countries. The Library is described in much detail in Bancerek et al. (2018).

5. From conceptual insights to formal proofs. Turing's O-machines and Gödel's idea of the inexhaustibility of mathematics

5.1. Insights, formalized proofs, algorithms, and mechanized proofs. Their mutual relations

The results of Gödel and Turing are complementary to each other with respect to the issue: how do creative intuitions give rise to mechanized formal proofs? Turing (1939) proposed a schema of an ordered sequence of ever stronger problem-solving machines, where each increase of algorithmic efficiency is due to a non-mechanical factor – called by him an "oracle". Any "ordinary" Turing machine equipped with Oracle is called an *O-machine*.

Commentators on this conception understand the activity of oracles as acts of mathematical intuition. Or, maybe, it should be rather said "philosophical intuitions concerning mathematical objects".

Gödel emphasises the role of philosophical intuition in mathematics, when claiming that the discovery of the incompleteness of number theory was due to his Platonic vision of objectivity and inexhaustibility (infiniteness of the domain) of mathematics.

Turing gives us a formal schema of ever stronger machines ordered in an infinite sequence, and does not pretend to state whether the human mind's cognitive abilities also allow proceeding in infinity; or maybe, at some point they would be too weak to attack the next, still more complex, problem.

Gödel is more optimistic in his hope that the frontiers of such a process might be pushed further and further by humans, also in difficult philosophical issues. He believed that many so-called philosophical problems are, in fact, scientific problems, only not yet examined by scientists.

He even tried to exemplify this claim, sketching a formal proof of Anselm's ontological argument for God's existence, so giving it a scientific form. After Gödel's death, this proof turned out capable of obtaining ever more precise form, owing to Dana Scott and other eminent logicians, up to the phase in which it has become possible to be processed by computer.

In that final phase, the project required mastering such sophisticated logical and computational measures as higher-order modal logics, and a combination of most efficient provers (Benzmüller, 2013, 2015). There is still another achievement in automating the ontological proof. It consists in controlling the correctness of the human handmade formalization, by using a program of the type *checker*, called also *proof assistant*. Such an approach was successfully adopted within the framework of analytic tableaux by Melvin Fitting (2002).[13]

Thus, a step has been taken towards realizing Gödel's idea, central to his optimistic rationalism, that is, his belief that people are able to perceive concepts more and more clearly, not only in mathematics but also in fundamental philosophy. For instance, the notion of most Perfect Being, conceived by Anselm in his ontological argument was becoming clearer and clearer in the successive reflections by Descartes, Leibniz, and Kant, up to Gödel, who was able to give it a strict logical form in an advanced logic.

Thus, it has been shown, at least in one question, that even in philosophy there can exist a way from insight to proof, not only formalized, but even computational. "Computational" means the highest degree of exactness and clarity, since every flaw will be detected by machine.

It ought to be noticed that it may also work the other way around. Not only from creative insights to formalized proof, but also from such proofs to new insights. Those, again, may push the frontiers forth, up to the next level of mechanization.

Benzmüller and Paleo (2013) remark that the exorbitant requirements imposed by automated procedures of problem-solving force the use of unusual logical means, e.g., some debatable systems of modal logic; otherwise the proof would not end with the conclusion we wish to get. This obliges us to reconsider the content of intuitions which motivate the logical system we use. Such a reflection may lead either to revising or to deepening these intuitions.

What Gödel and his followers did, formalizing the ontological proof, belongs to the discipline called *formal ontology*. What some of his followers did, those who devised provers or proof-assistants, can be called *computational ontology*. Such a discipline is being born before

[13] It would be impossible to list all relevant logical publication on formalizing ontological argument. A representative selection (ca. 40 items) is found in the Wikipedia entry "Gödel's ontological proof".

our eyes. This will, hopefully, open new perspectives on the issue of problem-solving with the united forced of creative insights and mechanical routines.

5.2. Oracles as non-mechanical devices to aid machines in solving problems which otherwise would remain unsolvable

There is a deep interconnection between what we call "computation" and "deduction". Each deductive step in a formalized system, that is, each move made by a problem-solving machine, is a kind of computation. In the case of deduction, this means computing the value of the consequence function. Hence, the question arises: is it possible to supplement a machine with uncomputable deductive steps? Such uncomputable steps in reasoning would be what we call "insights" or "acts of intuition".

Considering this question, Turing introduced the definition of an "oracle" which can supply, on demand, the answer to the halting problem for every Turing machine. Turing seems to have given this concept an interpretation in terms of a mathematician's "intuition" in theorem-proving. In fact, M. H. Newman in a biographical memoir on Turing identified the uncomputable "oracle" with intuition. This seems to go too far, as the "oracle" is capable of doing more than any human being. Nevertheless, Newman had a unique status as Turing's collaborator at this period and must have reflected the tenor of Turing's considerations. In any case, Turing in his definition of an oracle makes it clear that it enables one to see the truth of a formally unprovable Gödel statement.

The mentioned definition is contained in the passage opening Turing's article (1939). It runs as follows:[14]

> The well known theorem of Gödel shows that every system of logic is in a certain sense incomplete, but at the same time it indicates means whereby from a system L of logic a more complete system L' may be obtained. By repeating the process we get a sequence $L, L_1 = L', L_2 = L'_1, L_3 = L'_2, \ldots$ of logics each more complete than the preceding.

[14] Here it is quoted from the text of Turing's (1938) Ph.D. dissertation (1938), published in 1939. See URL: http://www.dcc.fc.up.pt/~acm/turing-phd.pdf.

To make this idea as accessible as would be needed by one for whom it would be unexpected, I avail myself of the notion of *essential extension*, opposite to what is known in logic as inessential extension. The latter is defined by Tarski, Mostowski, and Robinson (1968, p. 11) by two conditions concerning a relation between formal theories. The one relevant to the present issue, is the following: “An extension T_2 of T_1 is called *inessential*, if every valid sentence of T_2 is derivable in T_2 from a set of valid sentences of T_1. [...] If T_1 is axiomatic, then an inessential extension of T_1 is obtained by adding some new individual constants, but without adding any new non-logical axioms.”

When understanding “essential” as “not inessential”, we derive from the above text the following definition concerning axiomatic theories (just such ones as are considered in the context of mechanical problem-solving issues).

Axioms of a theory, besides their role of being first premises in proving theorems, perform the role of meaning postulates to define the content of concepts which occur in them, e.g. the concepts of zero and sequence in the axioms of arithmetic (see Definition 5.3 in Marciszewski [1981]). Note that in the process of creating a theory, such concepts are prior to axioms; only owing to the idea of zero conceived once upon a time, did it become possible for Peano to state his axioms. Such a process that leads to creating new axioms is worthy of being named *creative conceptualization* (cp. **0.1**).

If new logical axioms are added to the theory T_1, thus forming the theory T_2, then the latter is an *essential extension* of the former. Thus, all problems solvable in T_1 are also solvable in T_2, but not the other way round.

This terminological acquisition makes it easy to give a concise interpretation of Turing’s passage in the above textbox. To wit, that a system *L’* is closer to being complete than a system *L*, simply means that *L’* is an essential extension of *L*. The phrase “more complete” is to recall that e.g. the second-order arithmetic is closer to being complete than the first-order arithmetic, while the fully complete one is like the inaccessible limit of a sequence (Gödel, 1931, 1936).

Now it is easy to explain the role of an oracle. It is a means to advise such an essential extension of a theory, which is needed to solve the problem in question. As to the nature of the oracle, Turing does not go any further than saying that it cannot be a machine. With the help of the oracle we could form a new kind of machine, called an O-machine, having as one of its fundamental processes that of solving number theoretic problems unsolvable by ordinary Turing machines (Turing, 1938, p. 13).

Where do such insights come from? That is the question. Anyway, to be or not to be of scientific progress depends on the situations in which an oracle, that is, an enlightening insight, causes that a problem hitherto unsolvable, becomes solvable in a new and deeper perspective. From such insights are born also problem-solving machines, and those, in turn, assist us in getting new insights.

To incite a critical debate on the issue of intuition, I append a classical statement of mechanism due to Ludwig Wittgenstein as the author of *Tractatus Logico-Philosophicus*. In this way, those who oppose mentalism from the angle of mechanism win an opportunity to exactly define their mechanistic stance. Is it akin to that of Wittgenstein, or rather distanced from the philosophy of his *Tractatus*?

6. Wittgenstein's semantics and ontology: maxims on language and reality

6.1 Limits of my language mean the limits of my world

This means that all I know is what I have words for. Hence, what I cannot speak about, I must pass over in silence. It is a saying characteristic of Wittgenstein's semantical landscape. How famous it has become is witnessed by the dozens of its occurrences quoted in the German original, and almost 300 000 000 (!) in English translations (according to Google). In the original the maxim reads as follows:

5.6. Die Grenzen meiner Sprache bedeuten die Grenzen meiner Welt.

It is a view drastically inconsistent with whatever we know about language, mind and reality. We know that each mute animal has its own world consisting of what it perceives in the environment. We know that a very small boy acquires a native language in the process of perceiving some things, e.g., a black cat, and asking a parent: what is it? After listening to an answer, the child acquires the expression "black cat" which did not exist in his vocabulary before asking about the name. We also know that a discoverer of a new object or phenomenon, hitherto not known to anybody, proposes a name for it after the discovery, not before.

It is hard to believe that a great thinker could have thought something as counter-evident as the maxim in question. When looking for a broader context to explain the riddle, we find the following:

> 6.5. For an answer which cannot be expressed, the question too cannot be expressed. The riddle does not exist. If a question can be put at all, then it can also be answered.[15]

The riddle does not exist? Just a moment before, I felt it a riddle that in *Tractatus*, besides attractive and convincing ideas, sometimes appear such counterfactual views as item 5.6. Might the same author display so little sensitivity to such counterfactual evidence?

The answer may come by scrutinizing the words "can" and "cannot". There is impossibility, so to say, accidental, and another one – principal; the former removable after taking some steps, the latter insuperable. My riddle concerning Wittgenstein's insensitivity to some counterexamples is, I hope, just accidental.

As for the principal impossibility of answering, the paradigmatic cases were found, according to Wittgenstein, like the Vienna Circle, in metaphysics. Metaphysical issues were called by them *Scheinprobleme*, that is *pseudo problems*, having no chance to be solved by serious research. Such "riddles", they claimed, could not appear either in science or in scientific philosophy.

Suppose I am a neopositivist in the Wittgensteinian or the Vienna Circle style. Then in my language no answers to a pseudoproblem can be given because in a rigid language (the only I can accept) its rules do not admit concepts which are metaphysical, such as those of God, souls, universals, and even numbers, etc. – in accordance with item 6.5. Hence such notions do not belong to my language, being beyond the limits of my scientifically admissible world – according to maxim 5.6. Thus, that renowned maxim, freed from counterexamples appearing in ordinary language, refers to an ideal scientific language created according to that neopositivistic design.

[15] These claims are worth quoting in the original wording too: "Zu einer Antwort, die man nicht aussprechen kann, kann man auch die Frage nicht aussprechen. Das Rätsel gibt es nicht. Wenn sich eine Frage überhaupt stellen läst, so kann sie auch beantworten warden."

In the twenties of the past century, such a stance belonged to the mainstream of European philosophy. It embraced, as the fundamental principle, the belief in the decidability of logic and mathematics, that is, the *mechanical* (i.e., algorithmic) *solvability* of any problem arising in these and related disciplines. This conviction was undermined only in the thirties with the results of Gödel (1931), Turing (1936), and Church (1936).

Its clear-cut formulation by Wittgenstein is found in some passages which (in slightly differing wording) appear at several places in *Tractatus*.

> Our fundamental principle is that every question which can be decided at all by logic can be decided off-hand. [...] It is possible [...] to give at the outset a description of all "true" logical propositions. Hence there have n e v e r be surprises in logic. [...] Proof in logic is only a mechanical expedient to facilitate the recognition of tautology.[16]

This declaration sheds an additional light on the connection between limits of the world and those of our language. It is assumed that a right language should be based on the universal schema of predicate logic in which the whole mathematics, and related sciences, can be adequately expressed. As for logic (we read in the quoted passage) each problem is solvable in it with a *mechanical expedient*. The same is the case for mathematics because of the fact that the whole of mathematics is expressible in the language of logic. Solvability "off-hand" does not necessarily mean a quick solution, but that attainable in a finite number of steps (as said in the definition of algorithm), while "mechanical" means that no creative insight is needed.

Moreover, it is in order to note, according to the neopositivistic project of unified science that: (i) all sciences should be mathematized (ii) arguments in empirical sciences also will have mechanical (algorithmic) form, owing to the special *logic of induction* planned for the use of empirical sciences. In fact, so far this plan has not been realized, and even, as argued by Karl Popper, has no chance to succeed (1959).

Nevertheless, were this great project to succeed, Wittgenstein would be right in his claim: "for an answer which cannot be expressed,

[16] See in *Tractatus*: 5.551; see also 6.125, 6.1251, 6.1262; this item is interestingly pointed by Kneale (1962, p. 729); "never" italicized by Wittgenstein, "mechanical" underlined by W. M.

the question too cannot be expressed" since (in the paradigm of neo-positivism) the only right way of expression in sciences is in the algorithmized language of logic. Solely in terms of such a language it is possible to solve (algorithmically) any scientific problem. Hence, each problem is bound to be stated in the same terms in which one states its solution. In this sense, the solution can be expressed then and only then when the answer can be expressed.

This is the cornerstone of Wittgenstein's philosophy of language, mind and reality, as well as his philosophy of science. This foundation has been undermined at the most sensitive point, the belief in the decidability of logic and mathematics. This is the story told in the next section.

6.2. Wittgensteinian maxims in the context of logical atomism and of finitism

The term *logical atomism* is due to Bertrand Russell. However, it can be safely used as the name of Wittgenstein's doctrine too. He and Russell agreed that in the main features their philosophical views were concordant. This is why Russell so heartily welcomed Wittgenstein's *Tractatus* and preceded its edition (Wittgenstein, 1922) with an extensive and sympathetic introduction.

Logical atomism holds that the world consists of ultimate logical *facts*, *or atoms*, which cannot be broken down any further. It is often referred to as "Picture theory" for the tenet that the world is faithfully pictured by our language with the exactness 1:1 (one-to-one relationship). A map cannot consist of an infinite number of elements, hence the mapped world has to be a finite reality.

Let us consider implications of that approach for the issue of problem-solving. If one has such an ideal site map in which every detail of the site is mapped, then every question of how to get around in the terrain can be answered with the help of the map alone. Having been acquainted with the signs on the map, we are directly related to the corresponding objects in the terrain in question.

According to logical atomism, the right scientific language, describing a set of empirical facts connected by logical relations, supplies us with such an ideal map of reality. Thus, on the basis of the trustworthy mapping of the world by a properly constructed language, any problems concerning the world are *solvable* just with resort to the *logical analysis of language*. Since the only properly constructed language is that of classical propositional logic and predicate logic. This is why Russell's and Wittgenstein's atomism is honoured with the adjective "logical".

The above characterization of logical atomism, entailing the finiteness of the world, as well as the full cognitive availability of empirical facts and logical laws, leads to the conclusion that every scientific problem can be solved in a finite number of steps on the basis of the hitherto gained knowledge.

What is true about the above assertions is the fact that each of them is satisfied when it comes to the language of the propositional logic and its semantics. It is a language so closed that no new concept could enlarge the resources of its logical constants, beyond the combinatorially obtained number of twenty symbols. Thus, the limits of the language determine the limits of a conceptual apparatus concerning the references of logical constants which form the totality of the propositional domain ("world").

As for the item 6.5, the first and the third sentence amount to saying that if a question can be expressed, the answer can be expressed too. This is perfectly right about the language of propositional logic. Wittgenstein was convinced that the same has to be right about the language of whole logic, including the predicate calculus. However, this conviction has been refuted by Turing's and Church's sophisticated proofs of undecidability of that more advanced part of logic.

There is no chance to present here these highly technical arguments, but we can take advantage of a rough exemplification. Consider the following formula **CF** whose validity would be tested according to relevant rules of eliminating logical constants; to wit, rules belonging to the system of analytic tableaux.

CF: $\forall x \exists y (y > x) \Rightarrow \exists y \forall x (y > x)$ (CF stands for *curious formula*)

The structure of this formula causes that in the course of checking whether its denial (non-CF) does not lead to contradiction, our rules generate here the necessity of constantly repeated returns to the starting point. This process is carried out according to an invariably the same, perceived intuitively, principle of generating loops. Thus, after observing a number of steadily recurring loops we become certain of their inevitability which means that the process will never come to a halt. Thus, we solve in an intuitive way some case of the halting problem that in no case is solvable by Turing machine.

Thus, it is impossible to prove that non-CF leads to contradiction, i.e. to prove that CF is a tautology, that is, a logical truth. Shorter: the formula CF is unprovable.

Let's now take into account that, as Gödel (1930) has shown, the first-order predicate logic is complete. This means that whenever a formula is logically true, then it is provable. Hence, if it is not provable (as is CF), then it is no logical truth. So we have come to solve in an intuitive way a problem that is not mechanically solvable in the formalized language of logic.

Hence, contrary to Wittgenstein's stance, we come to the paradoxical conclusion that there are riddles which cannot be solved mechanically in an algorithmic language, but can be solved by an intellectual insight expressible in ordinary language. Were this point challenged by a defender of mechanism, such a rejoinder would be welcome as an encouragement to futher scrutiny.

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